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NEWTON'S *PRINCIPIA*.

THE short and troubled reign (1685–1688) of James II saw few additions to the domain of English literature, perhaps only Dryden's *The Hind and the Panther* being worth naming. To science, however, the three years mark an era in which the whole current of physical investigation was directed into new and spacious channels by the publication of *Philosophiae Naturalis Principia Mathematica. Autore JS. Newton. Trin Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali. Imprimatur S. Pepys, Reg. Soc. Praeses Julii 5, 1686. Londini. Jussu Societatis Regiae ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.*

Few copies of the first edition were printed, and the book was soon scarce: by 1691 we hear of a copy being bought for something over two guineas, a very considerable sum, since a good husbandman could be hired for three or four pounds a year: we hear also of the whole volume being copied out by hand. The second edition appeared in 1713, at the promptings of the great Bentley and under the editorship of Roger Cotes. The final edition, edited by Henry Pemberton, was published in 1726, just about a year before Newton's death. The best-known reprints are the so-called "Jesuit's edition", published at Geneva in 1739–1742 by Le Sueur and Jacquier, and the 1874 Glasgow edition of Thomson (Kelvin) and Blackburn. The first English translation was Motte's of 1729, and the latest the fine volume published by Cajori in 1934.

A great many bibliographical details about the *Principia* are to be found in H. Zeitlinger's essay, "A Newton Bibliography", in the *Newton Memorial Volume*, edited for the Mathematical Association by W. J. Greenstreet, and special reference to translations is made by E. H. Neville in his review of Cajori's book, *Gazette*, XIX, pp. 49–52, 134.

For the plate we are indebted to the courtesy of the Photographic Department of the British Museum.

A MATHEMATICAL APPROACH TO SOME GLASSWORKS PROBLEMS.*

By W. M. Hampton,

Technical Director, Messrs. Chance Brothers Ltd.

When your Secretary approached me first about a year ago with the request that I should discuss the application of mathematics in industry, I tried to take what I still feel would have been a wise decision, that is, I tried to refuse. Your Secretary, however, was so persuasive, and also suggested that it was the industry rather than the mathematics that you expected me to talk about, that I finally weakened. I am still amazed at my own temerity in getting up at a Society of Mathematicians to talk on this subject, particularly as I am not qualified in any mathematical sense. As far as I can understand it, mathematicians look on the subject as a language and almost an end in itself. If a problem can be resolved in mathematical terms, then that is the answer to the mathematician. Ordinary people like myself have as little appreciation of this attitude of mind as the average listener has of music. Mathematics to me is merely a tool that can be used as a help in solving a problem, and when the problem is solved in the sort of way that we hope, then the results must be translated back into ordinary words in order that they shall be of any use to us. In short, I use mathematics only as I might use a foot rule, or possibly a better analogy is a slide rule.

I felt I must make this clear from the start, as I should hate to keep you listening to me under false pretences.

The chief advantage of a mathematical approach to industrial problems in my experience is that before one can use mathematics, even in the very limited way I have indicated, one must be quite clear about the underlying assumptions one must make. I have always found it is necessary to make a great many simplifying assumptions, and this is doubtless partly due to my lack of knowledge of mathematics, as, if I do not simplify the problem to start with, then I get befogged in my maths and cannot solve the problems which I set myself. This may sound disappointing to you as a body, but I think it is necessary for one to make the matter clear, because even with this very limited application of mathematics to industrial problems I am looked on in the industry as a bit of a crank on this subject, and you will therefore realise the better how uninterested the average factory man is in the subject. I wonder if there is any implied criticism in the teaching of maths in this statement; I think there probably is, but I am not qualified to suggest a better approach—that is a matter for you teachers.

Even assuming that one accepts the very limited objectives I have put forward, valuable results can still follow from the method of attack I have suggested.

The late Dr. J. W. Mellor said: "One of the chief objects of scientific investigation is to find out how one theory depends on another; and to express this relationship in the form of a mathematical equation—symbolic or otherwise—is the experimenter's ideal goal." It is this shorthand method of writing the results of experience that makes the mathematical approach so interesting to me and so useful in industrial problems. I have many times told those whose work I have had to organise, that mere collection of facts is a sterile occupation; that unless these facts are collected in the light of a theory which it is desired to establish or disprove, then one is merely a collector of bricks and stones and one can never hope to be an architect. It is in the

* A paper to the Annual Meeting of the Mathematical Association, Birmingham, April 1949.

attempt to follow the implications of an assumption that I have found the most value in setting out a problem in mathematical form.

It is, of course, true that the start is always a fact, usually an awkward one, and the problem of the scientist is to produce a hypothesis that will connect this fact to others in a consistent manner and in such a way that a forecast can be made as to the behaviour of the material under various conditions. This expected behaviour can then be tested experimentally, and the hypothesis must stand or fall by the agreement, or lack of it, of the deduction with the experiment. A classic example is that of Newton and the falling apple. The awkward fact was brought violently to Newton's attention, and the theory evolved enabled all the discordant facts to be linked together. A further point may be noted here. The theory may not explain the fundamentals; Newton's gravitational theory does not explain gravitation; it merely says that the observed phenomena—from falling apples to the stately motions of the solar system—can be related on the assumption that all masses attract other masses in a manner which varies inversely as the square of the distances which separate them. This illuminating simplification enables forecasts of their future behaviour to be made with such precision that no one doubts that the time of sunrise for next August 1st as shown in our diaries will be exact—even if we can't see the sun on that day!

The bulk of the manufacture of glass at the present time is in glass tank furnaces. A tank furnace reduced to its simplest terms is a large bath open at the top to the flames from the burners, and so arranged that raw materials can be filled at one end while the melted, refined glass is withdrawn at the other end. This process operates, in general, continuously through 168 hours per week.

The sequence of operations involved in filling a quantity of cold, imperfectly mixed powdered material of non-uniform grading on to the surface of previously partially melted batch, and heating this by radiation and convection from the flame given by a variable fuel is sufficiently complex to make one pause. In addition, however, the molten glass is held in a refractory bath which is itself dissolving—although fortunately rather slowly—and the glass is finally needed to be free from bubbles, undesired colour, and variations in chemical composition; so the problem is indeed formidable.

Although the manufacture of glass in pots, which was the forerunner of continuous manufacture, is now becoming less important commercially, it is simple to start our consideration of the subject of this talk in terms of a pot furnace. In this case the process is discontinuous, which makes the mathematical approach simpler. It is known that when batch is melted the glass resulting from it does not correspond precisely with the calculated composition. This is due to the facts, amongst others, that:

1. The pot walls are dissolved into the glass.
2. That some of the constituents are volatile.

It is also known that the composition varies with the proportion of cullet added to the batch, *i.e.* previously melted glass. This, of course, is related to the two previous conditions in that cullet is glass which has been melted more than once and therefore tends to accumulate the effects of pot attack and volatilisation. Let us now apply a method of simple calculation to the problem of finding what the effect on the composition of the glass is on the process of melting in a pot. In order to simplify the problem even further, we will assume first that the glass is made entirely of non-volatile constituents. Stated mathematically, we require to find the concentration of a given constituent at any journey, *i.e.* run of the pot, from the commencement, knowing the original batch and the composition of the clay of which the pot is made,

and the proportion of cullet. We can also, without much risk, make the further simplifying assumption that the effect on the final concentration due to the total change in weight of the materials, caused by volatilisation on the one hand and dissolved clay on the other, is negligible. This is nearly true, even when we are taking volatilisation into account, since the total loss by volatilisation and the total gain on the solution of the pot are of the same order. We also make a number of other assumptions, possibly without meaning to, namely, that all the journeys are of equal length, that all the pots are run at a constant temperature, and also that freshly-melted glass from frit and re-melted cullet have sensibly the same rate of attack on the clay.

We start with the first journey being made entirely from batch, and subsequently with a constant proportion of cullet used in every journey. The following symbols are needed:

Let x = concentration of a particular oxide in the glass at any journey.

b = concentration of oxide in the glass as calculated from the batch.

d = concentration of the oxide in the clay of which the pot is made.

k = the proportion of the final glass which is contributed by the solution of the pot per journey.

c = the proportion of cullet used after the first journey.

The subscripts 1, 2, 3, etc., are used to indicate the particular journey.

Now, after the first journey:

$$x_1 = \text{concentration in the glass,}$$

and it is obviously equal to $b + dk$. Translated into words, the concentration in the glass is equal to the concentration from the batch plus the amount of the constituent contributed by the solution of $k\%$ of clay containing $d\%$ of the oxide. For the second journey x_2 equals the amount of oxide contributed by the batch plus the amount of the oxide contributed by the cullet plus the amount of oxide contributed by the clay, or in symbols:

$$x_2 = b(1-c) + cx_1 + dk,$$

which (since $x_1 = b + dk$) can be reduced to:

$$x_2 = b + dk(1+c).$$

Similarly for the third journey:

$$\begin{aligned} x_3 &= b(1-c) + cx_2 + dk \\ &= b + dk(1 + c + c^2). \end{aligned}$$

Hence, at the nth journey:

$$x_n = b + dk(1 + c + c^2 \ldots + c^{n-1}),$$

and by the rules of the summation of a geometrical progression this reduces to:

$$b + dk\left(\frac{1-c^n}{1-c}\right).$$

It follows that since c is less than 1, c^n becomes negligibly small as n becomes large, and after a large number of journeys the formula will become:

$$x_{n\to\infty} = b + \frac{dk}{1-c}. \qquad \text{(1)}$$

In this first calculation we have ignored volatilisation, and the problem becomes much more complex when allowance is made for it, since the concentration of each oxide is affected by the volatilisation from any other oxide in the glass. The calculation also becomes so complex for the intermediate stages that for the purpose of my argument I shall only refer to the steady state, that is, when sufficient journeys of constant batch and cullet proportions have been made for the glass composition to become constant. If v equals the loss of any oxide per unit concentration of that oxide per journey, and V equals the summation of all the losses by volatilisation, then by using similar methods to those indicated above the final concentration of the oxide in the batch is given by this rather cumbrous expression :

$$x = \frac{b(1-c)\left(1-\frac{v}{2}\right)+dk}{(1+k-V)-c\left(1-\frac{v}{2}\right)+\frac{v}{2}} \cdot \qquad (2)$$

You will see that if a particular constituent is non-volatile then $v=0$ and this expression can be simplified, and when, as was assumed in the earlier simple theory, $k=V$, the expression reduces to exactly the same as the one we had earlier. We will now apply this to an actual case.

The compositions calculated from the batch, the analyses of the clay of which the pot is made and of the glass run when a constant composition has been achieved using 40% of cullet, are given below :

	Composition calculated from batch.	Analysis of pot clay.	Analysis of glass when steady state reached.
SiO_2 - -	61·33	74·70	62·71
K_2O - -	12·55	0·50	12·00
Al_2O_3 - -	0·10	22·00	0·25
Fe_2O_3 - -	0·008	1·60	0·018
TiO_2 - -	0·012	1·20	0·020
PbO - -	26·00	—	25·00
	100·00	100·00	99·998

By solving the appropriate equations for the two non-volatile constituents SiO_2 and Al_2O_3, it can be shown that $k=\cdot0040$, that is, that ·4% of the final glass consists in effect of clay dissolved during any particular journey, and $V=\cdot1204$, or something like 12% of the constituents of this glass are lost by volatilisation each journey. In this particular case our simplifying assumption that $k=v$ is certainly not justified, but even so the nett effect on the calculation of the non-volatile constituents is relatively unimportant. In order to find the volatilisation of the individual volatile constituents, namely, potash and lead, it is necessary to use the figures from the analyses, but we still have two non-volatile constituents, *i.e.* iron and titania, whose expected proportions can be calculated on the basis of the formulae established, and the results are :

Iron - - - - ·019%

Titania - - - - ·020%

which agrees almost exactly with the analytical result.

Thus, by using this simple mathematical approach it is possible to deduce from a knowledge of the composition and two analyses the amount of clay dissolved each journey, and to show that the final composition is consistent with the change to be expected from contamination due to the clay. By doing a number of such calculations we have been able to establish factors which can be used for future estimates of the change in composition, both as regards volatilisation and pot attack, and instead of having to wait for a number of runs of different types of glass we can make quite reasonable estimates of what the composition will be before we start the manufacture. What, of course, is possibly more important is that the generalised knowledge indicated by these formulae enables estimates to be made of the effect of varying the proportion of cullet, of using a different sort of clay of different composition on the ultimate composition of the glass made in pots. In short, the use of mathematics cuts out a tremendous amount of expensive experimental work, and we have sufficient in the way of confirmation to show that the results are generally in accordance with such analytical results as we have obtained.

The case of continuous tank manufacture.

I shall assume that the tank has been running for some time and that a constituent is introduced in a new proportion. The symbols are essentially the same as for the pot problem except that, instead of referring them to a journey, they relate to a unit time and are relative to the total mass of glass in the tank. Thus :

x = concentration of a particular oxide in the glass being discharged.

a = the fraction of the tank content withdrawn in unit time.

b = concentration of oxide in the glass as calculated from the batch.

g = concentration of oxide in tank at start of experiment.

k = the weight of blocks dissolved in unit time relative to the tank content.

d = concentration of the oxide in question in the blocks of which the tank is made.

c = fraction of batch added as cullet.

It must now be assumed that the rate of block solution is constant with time, *i.e.* the amount dissolved is inversely as the throughput.

Then the amount of oxide removed by the glass being withdrawn is provided by the oxide added as batch, as cullet and by solution of the refractories. Again no allowance is made for volatilisation.

The nett change in concentration ∂x is then for a small interval of time ∂t given by :

$$\frac{\partial x}{\partial t} = ab(1-c) + acx + kd - ax,$$

and integration of this yields the following :

1. When the oxide is originally absent from the tank :

$$x = \left[b + \frac{kd}{a(1-c)}\right][1 - e^{-a(1-c)t}]. \qquad \text{(3)}$$

2. When the oxide is originally present in concentration g :

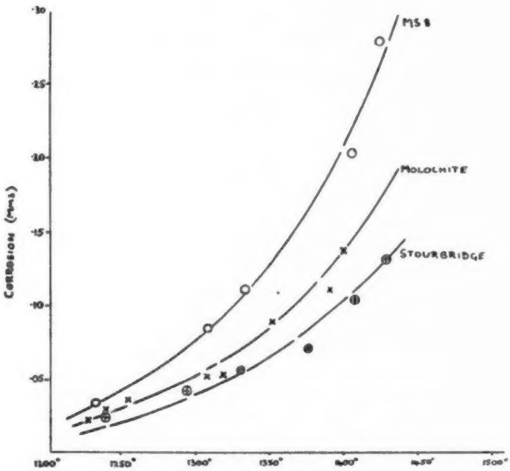
$$x = b + \frac{kd}{a(1-c)} + \left[g - \left(b + \frac{kd}{a(1-c)}\right)\right][e^{-a(1-c)t}]. \qquad \text{(4)}$$

As t increases and conditions become steady, then in both cases :

$$x = b + \frac{kd}{a(1-c)} \cdot \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The similarity of (5) to the expression for pots is obvious, and it may be deduced that the concentration of an oxide which is derivable from the blocks decreases as the throughput increases and as the proportion of frit increases. Further, the concentration in the glass remains constant when the product $a(1-c)$ = amount of fresh material added is constant. In practice, then, one could if desired compensate for changes in throughput by altering the cullet proportion.

The diagrams (Fig. 1) show the results of varying the proportions of cullet for a constant throughput and of varying throughput for a constant cullet proportion.



TEMPERATURE 0° C.

FIG. 1. Corrosion tests with Dedf. 748278, Corrosion v. Temperature.

The values assumed are :

$b = \%Fe_2O_3$ in batch $= 0{\cdot}021$

$k = 0{\cdot}0019$/day. $\quad d = 1{\cdot}0\%$

Constant throughput $\quad a = 0{\cdot}28$

Constant cullet $\quad c = 0{\cdot}50$

As in the pot case, the allowance for volatilisation is complex, but the problem has been solved, although the results will not be of interest to the present audience. An example, however, will show the kind of result that may be expected.

	Composition calculated from batch.	Analysis of blocks.	Analysis of final glass.
SiO_2 - -	72·58	55·20	72·72
CaO - -	10·50	—	10·51
Na_2O - -	16·50	0·74	16·27
Al_2O_3 - -	0·14	41·50	0·25
Fe_2O_3 - -	0·014	1·36	0·018
TiO_2 - -	0·014	1·20	0·016
As_2O_3 - -	0·25	—	0·22

The values of the constants were (time unit—1 week)

$$a = 2{\cdot}50,$$
$$c = 0{\cdot}40,$$

and the values of k and V deduced were

$$k = 0{\cdot}00398, \quad V = 0{\cdot}00385.$$

The extremes of composition obtainable are :

	$a = 0{\cdot}50$	$a = 3{\cdot}00$
	$c = 0{\cdot}80$	$c = 0{\cdot}10$
SiO_2 - -	74·10	72·67
CaO - -	10·40	10·50
Na_2O - -	13·52	16·37
Al_2O_3 - -	1·78	0·20
Fe_2O_3 - -	0·068	0·016
TiO_2 - -	0·062	0·016
As_2O_3 - -	0·07	0·23

It will be seen that the range of final compositions can thus vary nearly 3% in soda content from the same tank with a constant frit composition, due solely to changes in technique, and the figures give an indication of the importance of continuity and uniformity of procedure.

You see why glass manufacturers find week-ends and holidays a trial!

Turning now to the second problem, I would like to discuss the corrosion of the side wall blocks. These frequently come from a tank with such a sharp cut at the glass level that it has been suggested this is due to some specially corrosive constituent which floats on the surface of the glass.

Before one could accept this it would be necessary to establish that there was such a corrosive layer, and secondly that the attack could not be explained more simply by the normal solution of the tank block by the glass. Now it is well known that in any glass tank there must exist a marked temperature gradient through the depth of the glass. We are not concerned at the moment with how this gradient arises, but merely with the fact that it does exist. Halle * has published measurements for the temperature gradient through ten different glasses in large tank furnaces, and has compared them with the gradients through the same glasses in a small-scale tank.

The mean gradient through the small tank was much greater than it was through the deeper tank. This was such an unexpected result that a paper recently published by C. E. Gould,† which was stimulated by the surprising nature of the statement, has shown that in fact there is no inherent inconsistency in the two cases, but that one curve is simply an extension of the

* *J. Soc. Glass Tech., Trans.*, 1947, 31, 122–133.

† Ibid., 213.

other, and as the curve is not linear then the mean gradient over a small range in the small tank is necessarily greater than the mean range in the bigger tank. In the table, the results of Gould's calculations are given for the temperatures to be anticipated at different depths in a tank 24 inches deep, based on the experimental observations of R. Halle and W. E. S. Turner for a small-scale tank with a glass depth of 5 inches. The calculated and observed figures are in good agreement.

Soda-lime-silica glass: Iron oxide 0·04 %.

t (inches.	Measured *T*.	Calculated *T*.	Difference.
0	1450°	1450°	—
0·5	1400	1394	− 6
1·0	1381	1381	0
2·0	1355	1370	+ 15
4·0	1334	1355	+ 21
6·0	1324	1345	+ 21
10·0	1310	1330	+ 20
14·0	1302	1320	+ 18
18·0	1298	1311	+ 13
23·0	1292	1302	+ 10

As F. W. Preston has said in a private letter to me on this point, this must necessarily be so because the heat does not know whether the tank is going to be shallow or deep until it gets through. Anyhow, as a result of these measurements and calculations it is now possible to make a reasonable estimate of the temperature gradient down a glass tank, being given a knowledge of the iron content of the glass in it. Now let us make the assumption that the blocks are dissolved away at a rate proportional to the rate of solution of that refractory in that glass at a temperature which exists at any point. Such measurements of the temperature-corrosion curve for a given association of glass and refractory can easily be made in the laboratory, and a typical curve is shown. (Fig. 2.)

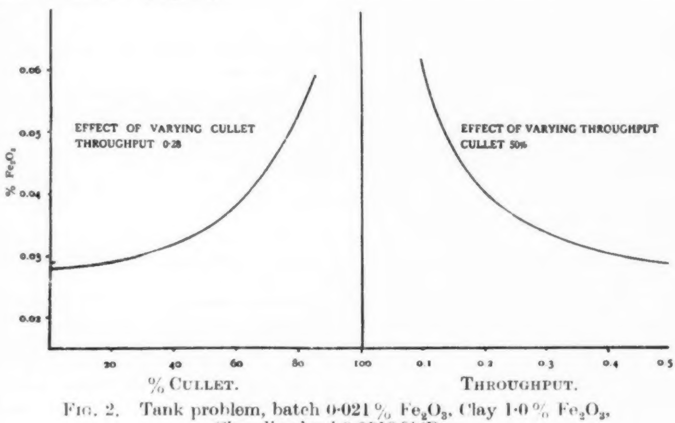
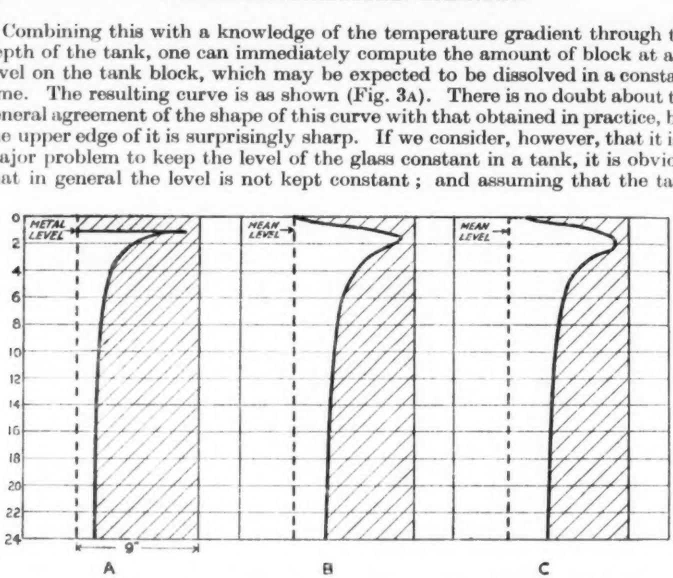


FIG. 2. Tank problem, batch 0·021 % Fe_2O_3, Clay 1·0 % Fe_2O_3, Clay dissolved 0·0019 %/Day.

Combining this with a knowledge of the temperature gradient through the depth of the tank, one can immediately compute the amount of block at any level on the tank block, which may be expected to be dissolved in a constant time. The resulting curve is as shown (Fig. 3A). There is no doubt about the general agreement of the shape of this curve with that obtained in practice, but the upper edge of it is surprisingly sharp. If we consider, however, that it is a major problem to keep the level of the glass constant in a tank, it is obvious that in general the level is not kept constant; and assuming that the tank



Shape of block when thickness at point of maximum attack is 1″; metal level constant.

Shape of block when thickness at point of maximum attack is 1″; metal level varied—standard deviation ±0·5″.

Shape of block when thickness at point of maximum attack is 1″; metal level varied—standard deviation ±0·75″.

Effect of random variations in level on the shape of the block.

FIG. 3.

level at any moment is given by a random variation about the mean value, one can compute what the level of the glass is as a proportion of the total time the glass and refractory are in contact, in terms of any assumed standard deviation in the level of the glass. When the tank level falls, then the whole temperature down the tank is raised slightly, and when the tank level rises the temperature falls slightly. By combining this variation of level with the previously computed curve for the solution of the tank block as a function of time, one can compute the probable shape of the block for this tank as a function of the variation in level.

The diagrams (Figs. 3B, 3C) indicate the effect of this random variation in level, and the shapes so computed are really surprisingly like the shape of used tank blocks with which the industry is familiar.

In short, the shape of used tank blocks is quite consistent with the expected solution of the block by the glass, and there is no need to pre-suppose that any peculiarly corrosive layer occurs at the surface. Thus, by the use of a few simple assumptions it is possible to show what the shape of a tank block might be expected to be, and, what is more, to deduce this shape from a knowledge of the iron content of the glass, which can be largely calculated from the batch, and from straight laboratory experiments of the rate of

solution of any type of tank block in the particular glass being used. But one can go further. It has been said that the life of a tank is the life of its flux-line blocks. In other words, a tank has to be put out when the glass has penetrated at the point of worst attack, which is of course the flux line. If the level varies substantially, then instead of having a sharp, knife-like cut at one particular point, this cutting action is spread over a greater area and the depth of penetration in a given time is much reduced. It follows that the accurate control of glass level must tend, other things being equal, to reduce the life of a tank by dissolving the blocks at the flux line ; and if it is necessary to control the level of a tank fairly precisely, as it is for many modern machine operations, then an increased life can be attained by running the level at a precise figure, but making specific and definite changes in level from time to time. The table gives the relative tank life for various assumed conditions of variation in level without any alteration of glass, blocks and temperature.

RELATIVE LIFE OF TANK BLOCKS AND STANDARD DEVIATION OF THE GLASS LEVEL AT THE FLUX LINE.

Standard deviation of level (inches).	Relative life.
0	1·00
±0·25	1·45
±0·50	1·80
±0·75	2·08

It cannot, of course, be taken as a precise indication of how long a tank will run, because of the many variables which operate. One of the most important of these is the question of commercial convenience, since a tank often has to go out for repair because it is convenient to make a repair at that time, whereas if it were vital to keep it going it could run on for a substantially longer period. The table does, however, give an indication as to the variation which may be expected in life apart from these commercial considerations, if it is operated on a precise and predetermined schedule.

It may fairly be argued that there is no mathematical work in this example, and that is in one sense true, but I submit that the whole approach is mathematical in its implication and as such is permissible under my general heading.

The thermal performance of glass tank furnaces.

I propose now, as my third example, to discuss the derivation of a formula for the comparison of the thermal performance of different tank furnaces.

The fuel consumption of a furnace is clearly a function of many variables. The chief ones are :

1. The temperatures at which the various compartments of the furnace are run.
2. The size of the furnace.
3. The quality and type of the fuel used.
4. The rate at which glass is melted and drawn from the furnace.

The quantity of glass made, in turn, depends on :

5. The standard of quality required.
6. The chemical composition of the glass.
7. The manufacturing requirements, *e.g.* size of article, rate of production, etc.

Now some of these are inherent in the furnace and some are imposed by

commercial requirements. For example, the size is fixed, but the rate of melting is to some extent imposed by the Sales Department's needs.

In order to allow comparisons to be made between different furnaces with any accuracy, the performance figure must be free from variation for these imposed factors so that, for example, the same performance figure must be derived for the same furnace, irrespective of the operating temperature, glass composition or rate of output.

In the formula put forward by the Furnace Committee of the Society of Glass Technology,* this was achieved by applying corrections to the actual consumption of heat units by allowing for :

1. Departure of size from a standard.
2. Departure of actual temperature from a standard.
3. Correcting the fuel consumption to a no-load value.

It was realised that glass colour and the age of the furnace had an effect on fuel consumption, but no data were at the time available to allow corrections to be applied.

The fundamental assumption was made that the function over which the designer had control was the capacity of the furnace to reach and maintain a pre-determined temperature over the working region, and that the most efficient furnace was the one which performed this function with the least expenditure of fuel.

Thus, Q was defined as the number of B.Th.U's required per second to maintain 1 sq. ft. of melting area at 1400° C. when the furnace was not melting glass.

The formula for the derivation of Q is :

$$Q=\frac{0\cdot026}{AK_A}\left[\frac{GH}{K_T}-Wc\right], \qquad (6)$$

where :

		Unit.
G =	Total quantity of heat consumed/24 hours	tons/24 hrs.
H =	Heat content of fuel at the valve	B.Th.U./lb.
A =	Area of the glass in the furnace exposed to the crown	Sq. ft.
W =	Weight of glass/24 hours	tons/24 hrs.
c =	Quantity of heat to melt the materials and raise to 1400° C./lb.	B.Th.U./lb.
K_T =	Correction factor for temperature.	
K_A =	Correction factor for area.	

Now this formula has proved to be of the utmost practical value to the glass industry, and has enabled manufacturers to compare their own furnaces with others, to study the effect of furnace age on efficiency, and to study the effect of glass colour on fuel consumption.

To-day, however, I want to show how the two vital factors K_T and K_A were reached by theoretical reasoning.

Derivation of K_T.†

The heat input at the inlet of a furnace is either absorbed or lost in three ways :

1. That used for glassmaking, *i.e.* the heat used in the necessary thermal and chemical reactions and in heating the body of the glass, heat which is finally dissipated when the glass is removed from the furnace.

* *Trans. J. Soc. Glass Tech.*, 1944, p. 33.

† *J. S. G. T.*, Hampton, 1941, p. 249 *et seq.*

2. That communicated to the walls of the system, which heat is subsequently lost by radiation and convection.
3. That lost up the stack in the waste gases.

It should be noted that for the standard conditions, *i.e.* when the balance is taken, the effect of the regenerators can be ignored in this calculation, since the heat added to the incoming gas (and air) is taken from the heat which would otherwise go with the waste gases up the stack. Thus in general :

Heat input to valve = Heat used for glassmaking + heat lost through walls + heat lost up stack.

If H_1 = heat input to furnace from fuel supply,
H_2 = heat used in glassmaking,
H_3 = heat lost from hearth walls,
H_4 = heat lost from regenerator walls,
H_5 = heat lost up stack,
S = heat recovered by regenerators.

Then,

$$H_1 + S = H_2 + H_3 + H_4 + H_5 + S$$

or

$$H_1 = H_2 + H_3 + H_4 + H_5. \qquad (7)$$

The effect of temperature changes and glass load on heat required at valve.

Since no published data relating to the effect of temperature on fuel consumption have been found, it is necessary to derive a relation on theoretical grounds which will allow such a correction to be applied. It is clear that any such relation can only be approximate, since it will vary from furnace to furnace.

In order to simplify the calculation, it is convenient to idealise the furnace-conditions. Thus, it is assumed that a single-compartment tank furnace is run at a uniform temperature of $T_0°$ for the condition when the balance was measured. It is assumed also that the external walls of the hearth are at a uniform temperature of $t_0°$.

If the glass load (that is, the weight of glass discharged from the furnace per unit of time) is now altered so that the amount of heat required for glass-making is nH_2, then :

$$n = \frac{\text{glass load under revised conditions}}{\text{glass load under standard conditions}} = \frac{\text{new load}}{\text{standard load}}.$$

If the temperature of the tank furnace and of the external walls are now modified so that the temperatures are T and t instead of T_0 and t_0, a new value of the fuel input will be necessary.

Let $$R = \frac{\text{heat input required for temp. } Tt \text{ and load } n}{\text{heat input required for temp. } T_0 t_0 \text{ and } n = 1}.$$

Consider each of the items H_1 to H_5 in turn. The new input is by definition RH_1.

The heat used for glassmaking consists of two parts :

1. The heat used in heating the glass to the temperature at which it is withdrawn from the furnace.
2. The heat used in the chemical reactions to produce glass from raw materials.

The second item is small, and has been ignored for the purpose of the present calculation.

Assuming that the specific heat of the glass does not change over the range

of temperature involved, then the new value for the heat used for glassmaking is :

$$nH_2 \frac{T}{T_0}.$$

The heat lost through the walls of the hearth and regenerators is a complex summation of terms of similar form.

Considering the hearth loss, it is clear that since the heat lost is proportional to the temperature difference between the inside and outside surfaces, the loss under the new conditions will only be a function of the new temperatures, and will not involve R or n. In other words, the new loss from the hearth will be :

$$H_3\left(\frac{T-t}{T_0-t_0}\right).$$

In the case of losses from the regenerator walls, it is clear that these depend on the temperature gradient through the wall. The inside temperature will increase with rise of furnace temperature, since the gases passing in will be at a higher temperature, and in general may be written kT. The outer temperature will also rise with the inner temperature, and as an approximation is also taken as kt. If the supply of gas to the regenerator increases, these temperatures will rise due to that cause also. Thus the loss may be written as :

$$RH_4\left(\frac{k(T-t)}{k(T_0-t_0)}\right)=RH_4\left(\frac{T-t}{T_0-t_0}\right).$$

Over the range of temperatures considered, it is found that the ratios

$$\frac{T}{T_0} \quad \text{and} \quad \frac{T-t}{T_0-t_0}$$

are so nearly equal as to allow the simple value $\dfrac{T}{T_0}$ to be used for both, within the limits of accuracy of the other assumptions.

If the basic equation be considered in general terms, using affixes for the new values of the various heat items, then :

$$H_1'=H_2'+H_3'+H_4'+H_5', \quad \text{.................................(8)}$$

where

$$H_1'=RH_1,$$

$$H_2'=n\frac{T}{T_0}H_2,$$

$$H_3'=\frac{T-t}{T_0-t_0}H_3=\frac{T}{T_0}H_3 \text{ approx.},$$

$$H_4'=RH_4\frac{T-t}{T_0-t_0}=RH_4\frac{T}{T_0},$$

H_5' can be eliminated as follows :

For the conditions of the original balance the heat leaving the combustion space is H_4+H_5+S.

If the temperature and throughput of gas be changed, the heat leaving will be $R\dfrac{T}{T_0}(H_4+H_5+S)$.

In general terms this is :

$$H_4'+H_5'+S', \text{ where } H_4'=R\frac{T}{T_0}H_4,$$

whence :

$$H_5' + S' = R\frac{T}{T_0}(H_5 + S). \qquad \text{(9)}$$

The heat entering the furnace

$$\begin{aligned} &= RH_1 + S' = H_2' + H_3' + H_4' + H_5' + S' \\ &= \frac{T}{T_0}[nH_2 + H_3 + RH_4] + H_5' + S' \\ &= \frac{T}{T_0}[nH_2 + H_3 + R(H_4 + H_5 + S)]. \end{aligned}$$

If the regenerator efficiency $= E' = \dfrac{S'}{S' + H_4' + H_5'}$, ...(10)

then

$$S' = RE'\frac{T}{T_0}(H_4 + H_5 + S) = \frac{E'}{E}RS.$$

Substituting :

$$R = \frac{\dfrac{T}{T_0}(nH_2 + H_3)}{H_1 - \dfrac{T}{T_0}(1 - E')(H_4 + H_5 + S)}. \qquad \text{(11)}$$

This gives the value of the heat input required relative to the value at which the balance was made, in terms of the glass load, temperature and the regenerator efficiency.

For the purpose of finding the relation between R and T it is necessary to consider a balance sheet. The essential data which were used for the purpose are given in table below.

THE EFFECT OF TEMPERATURE CHANGES ON THE HEAT REQUIREMENTS OF A GLASS TANK FURNACE.

	Heat input and distribution at original temp. of 1450°.
H_1 - -	100·0
H_2 - -	9·2
H_3 - -	45·5
H_4 - -	19·8
H_5 - -	25·5
S - -	37·0
E - -	0·45

By similar calculations the values of R, found relative to unit input at 1400° for the temperatures below, are respectively :

1·457 at	1700° C.
1·133	1500
1·000	1400
0·881	1300
0·674	1100

Derivation of K_A.

From the basic equation of heat exchange it is clear that the heat put into the furnace is used in losses through the walls and crown of the furnace and in the gases to the stack.

If the gas composition and proportion of air remain constant, with a constant pre-heating, it may be assumed that a constant proportion of the heat input reaches the furnace chamber. If the temperature of this chamber remains constant, then the heat removed in the waste gases also remains a constant proportion.

For two similar tanks of different dimensions the heat lost through the crown is proportional to the area of the tank. Since a " no-load " condition has been postulated, the heat conducted or otherwise transmitted through the glass is also proportional to the area. The losses through the side walls are, however, proportional to the linear dimensions, since all tanks have nearly the same height.

Thus the losses through the walls of the furnace chamber can be expressed by a function of the form : $ax^2 + bx$, where x = linear dimension of the tank, *i.e.* length of side. a and b are constants.

This must be a constant proportion of H_1, the heat input, or in general terms :

$$H_1 = ax^2 + bx. \qquad (12)$$

Now a normal size of tank furnace has an area of 600 sq. ft., so that it was decided to correct to this size.

Now " Q " is defined, under particular conditions, as the heat required per unit area, or :

$$Q = \frac{H_1}{x^2} = \frac{ax+b}{x}.$$

Whence :

$$H_1 = (ax + B)x.$$

An examination of published data enabled an estimate to be made of the values of a and b, and the relative value to that required for a furnace of area 600 sq. ft. is given by the following :

$$K_A = \frac{A + 60\sqrt{A}}{3{\cdot}45A}, \qquad (13)$$

since

$$A = x^2.$$

Detailed figures for the consumption of a whole range of industrial furnaces is given in the *Journal of the Society of Glass Technology*, 1941, p. 258, and from these the values of the performance figure (eqn. 6) have been worked out, but without applying a correction for the size of the furnace. These results are shown in the next slide, plotted against the area of the furnace. The shape of the curve is that deduced from the theory developed above, and it will be seen that this covers the experimental points reasonably. The next slide shows the performance figures corrected for area, and they then form a random distribution about the mean value of 6·1. This, it seems to me, is a justification for the form of the correction factor, and the residual differences in the performance then give a true measure of the thermal efficiency of the different furnaces in operation. Having data of this kind, which enables furnaces of different sizes, making different glass and run at different temperatures, to be compared on a uniform basis, gives an opportunity of studying the factors of operation and design which control the thermal efficiency.

A number of papers have been published in recent years on the use of this formula, and it is now fairly established as a most useful and reliable guide to the estimation of furnace efficiency, and yet it is based on nothing more than plausible assumptions, clothed in mathematical forms, and is, I think, the best example I can show you of the value of the method of approach I have had the privilege of discussing before you to-day. W. M. H.

THE GEOMETRY OF MANY DIMENSIONS.*

BY J. L. SYNGE.

Dimensional possibilities.

I hear people complaining about high prices, scarcities, world unrest, and even the weather. I feel inclined to say : " Foolish people, you should be on your knees thanking God that it was into a world of three dimensions He put you, instead of into one with only one or two! "

Have you ever considered how dull it would be in a world with only one dimension? Life would be just one long queue. We would be like beads strung on a thread. This lecture would be impossible, because the first member of the audience would hide the lecturer from all the rest. It would be an intolerable life unless relieved by the multiple connectivity we find in railway lines and electric circuits. But even with multiple connectivity I cannot think that one dimension is enough.

Two dimensions would be a little better, but still dull. By looking over one another's heads (as you are doing at present), you could all see the blackboard. But just as the dimensionality of this blackboard on which I write is $3-1=2$, so in a two-dimensional world the dimensionality of the blackboard would be $2-1=1$. I could use it only for making dots and dashes. In the city the traffic could not keep to the left, because there would be no left. One vehicle could get past another only by going over it or under it. And for pedestrian traffic there would have to be an elaborate etiquette as to who should jump over whom.

Now in three dimensions—but I need not tell you about three dimensions. You know how good life is in three dimensions. If you get run over in College Green, do not blame the three-dimensional world—blame the Corporation that recognises only two. Or buy yourself an aeroplane or a pair of stilts.

But it is the nature of man to be permanently dissatisfied. If I continue to extol the merits of three dimensions, you will become bored and say : " Why not four? " Yes, indeed, why not four? Or five? Or six?

Now when I speak of a world of four dimensions, I am not thinking of slipping in time as a fourth dimension. That is another story altogether. I mean four *spatial* dimensions. What would it be like to live in a world with four spatial dimensions?

First for some broad generalities.

At present our clothing is essentially a two-dimensional affair. The Romans favoured simple connectivity, the Gauls (and ourselves) multiple connectivity. In a four-dimensional world, two-dimensional clothing, whether simply or multiply connected, would satisfy the requirements neither of warmth nor decency. We would need three-dimensional clothing to cover the three-dimensional skins of our four-dimensional bodies.

But you would not add one to the dimensionality of everything. At present we write on two-dimensional paper, but writing is essentially a one-dimensional operation, the letters being queued up one after another. In a four-dimensional world we might use three-dimensional letters, but they would have to be arranged in order—that is to say, writing would be essentially as one-dimensional as it is now, because ordered thought would (I presume) remain one-dimensional. Similarly dinner menus would remain essentially one-dimensional, even if you were drinking four-dimensional wine out of a glass that tried to be as three-dimensional as possible.

But these are matters of no interest to mathematicians. Let us get down to business.

* Address delivered at the Opening Meeting of the Dublin University Mathematical Society, Feb. 7, 1949.

From intuition to definition.

How are we to begin to think in four dimensions? I can offer only one plan. Think hard in all sorts of ways about the familiar things in three dimensions. At each new approach ask yourself: Can this method of thinking be carried over from three to four dimensions?

We are all human beings long before we are mathematicians. If someone says the word " box ", we think of a box in a way human beings think of it. That is to say, it summons up a complex of imaginary reactions of sight and touch. We know what a box looks like and we know what a box feels like. In brief, we have what might be called an intuitive concept of a box.

Suppose someone asks us: " How many corners, edges, and faces has a box? " We say at once (or perhaps after a moment's thought) that a box has 8 corners, 12 edges, and 6 faces. Perhaps you remember these numbers. I don't. It's a good thing I have them written down here, because otherwise I might get muddled before this audience.

Now how do I find out those numbers? You say " box ", and at once in my mind there floats the image of a box, semi-transparent so that I can see through it. Then I count the corners, edges and faces, running a mental finger over them to check them off if I get confused.

Perhaps your method is a little different. But I am pretty sure that you do not mentally write down the equations of the faces and go through the algebra of their common solutions. In fact, I am pretty sure that you, like me, behave like a human being when faced by the word " box ".

I should say rather that you and I behave like three-dimensional human beings in this case. Our intuitions of sight and touch are essentially the intuitions of three-dimensional men and women—or perhaps I should say " children ", for by the time we have grown up we are usually too dull to get really vivid intuitions.

That, then, is one way of thinking about a box. As far as extension into four dimensions is concerned, I will not say that it is wholly useless, but it is not much good. The best thing is to become very humble and simple and start at the beginning.

Instead of taking a complicated thing like a box, let us go back to the simplest geometrical thing—a *point*.

Now, in this lecture, I am not trying to put a fine logical edge on things. I assume that you have a good practical knowledge of ordinary geometry, and I am trying to carry over that kind of knowledge into four dimensions. There are several ways of doing this. I am going to use the method which is quickest and most reliable—namely, the algebraic method.

It is well known that mathematics is a deductive science—you prove things. But it must be realised that mathematics is also a descriptive science—it sets up names and definitions to describe mathematical situations. It is also an inductive science—it passes from the particular to the general. This last is not a feat of logic—it is a feat of imagination. Such feats of imagination appear difficult at first, but very simple when you look at them in the proper light.

The dictionary.

We are now going to prepare a dictionary. There are four columns. The first is headed " Thing ", and the other three have the general heading " Dimensions ", and under that, " 3, 4, n ".

Thing	*Dimensions*		
	3	4	n

I have not mentioned n dimensions previously. It is a strange fact that once we break out of the confines of three dimensions, it is almost as easy to go on to n dimensions as it is to stop at 4. Indeed, I would not be surprised if in two hundred years or so schoolboys will start with the geometry of n dimensions and get the results in plane geometry by putting $n=2$. The only danger in such a proceeding would arise from careless division by a factor $(n-2)$, which would be quite all right for values of n different from 2, but might lead to the most extraordinary results when $n=2$.

But to return to our dictionary.

I fill in the first line like this :

Point	(x, y, z)	(x, y, z, w)	$(x_1, x_2, \ldots x_n)$

Whatever other thoughts you may have about a point in three dimensions, you must admit that it has a unique triad of coordinates (x, y, z) for given rectangular axes, and that this triad defines that point alone. In fact, the triad *is* the point.

This is a way of thinking about a point which can be easily carried over into 4 or indeed n dimensions. In the case of four dimensions, we just stick in a fourth coordinate w, and we say that a point in four dimensions *is* a tetrad of numbers.

Having thus dealt with four dimensions, anyone can see what we should do for n dimensions : define a point as a set of n numbers. But the ordinary notation gets clumsy, so we write the n coordinates with subscripts,

$$(x_1, x_2, \ldots x_n).$$

Straight lines.

After " point ", I suppose the next thing to define is the straight line. I shall make the following entry in the dictionary :

Straight line determined by two given points	$x=ax'+bx''$ $y=ay'+by''$ $z=az'+bz''$ $a+b=1$	$x=ax'+bx''$ $y=ay'+by''$ $z=az'+bz''$ $w=aw'+bw''$ $a+b=1$	$x_1=ax_1'+bx_1''$ $x_2=ax_2'+bx_2''$ .... $x_n=ax_n'+bx_n''$ $a+b=1$

In making up this entry, the first thing is to throw the equations of a straight line determined by two points in three dimensions into a form easy to generalise. I have chosen one of the standard forms. Here the given points are (x', y', z') and (x'', y'', z'') ; a and b are variable parameters bound by the relation $a+b=1$.

To generalise to four dimensions does not require much imagination. Now the given points are (x', y', z', w') and (x'', y'', z'', w''). In four dimensions we *define* the straight line determined by these two points by the equations I have written down.

Now if you are not drugged by the beauty of these algebraic formulae, you should at this moment protest. You should say something like this : " Here is a new-born baby—a set of algebraic equations. You had a name all ready, and you have given it to this baby. How do you know that there are not other babies more worthy to receive the honourable name of ' straight line determined by two given points ' ? " My first impulse is to answer rudely : " Go and find another baby if you don't like this one." But if I am more patient I shall say that this baby has won my heart by its simplicity. It is true that I have in theory a perfectly free hand in defining a straight line in four dimensions, but the freedom is actually illusory. Mathematicians are

constantly asserting freedoms which they have no intention of exercising, unless perchance there happens to be a Ph.D. candidate in search of a problem.

I need not linger over the definition of the straight line in n dimensions; its structure is obvious.

This business of generalisation may appear an easy and amusing occupation. So it is. But there is a source of worry which is always present to the conscientious mind.

We are taking ordinary words—point and straight line—and using them in a new extended sense, which we have clearly defined. But, being human, we cannot kill the old sense entirely. If we are a little tired or sleepy, we find ourselves attributing to points and straight lines in four dimensions properties that they have in three—properties so simple and obvious that we hardly realise that they might not exist. Perhaps these properties hold in four dimensions also, but it is a shock to think that we might be carelessly assuming something false.

Here is an example : *The straight line determined by two given points contains those points. True or false?*

In our ordinary geometry we would probably never ask this question. Of course the straight line determined by two points contains those points. Just look at it!

But in the unfamiliar territory of four dimensions we lack the friendly guiding hand of intuition. Every word we use must be sifted carefully. In the case of the above question the answer is *true*, and you can see it in a moment. But such things may give you a nasty jar from time to time. Nevertheless, I feel that there is something to be said for living dangerously in mathematics, and accepting some things without proof on the basis of instinct. Much modern mathematics concerns itself so much with fine detail that the general pattern is lost to all but the specialist.

Distance and the triangle inequality.

We have now got points and straight lines in four or n dimensions. Since you also have the idea of the dictionary, I can go a little faster.

The distance between two points is the next thing to define. I write the formulae :

3 dimensions : $\sqrt{\{(x'-x'')^2+(y'-y'')^2+(z'-z'')^2\}}$,

4 dimensions : $\sqrt{\{(x'-x'')^2+(y'-y'')^2+(z'-z'')^2+(w'-w'')^2\}}$

n dimensions : $\sqrt{\{(x_1'-x_1'')^2+(x_2'-x_2'')^2+\ldots+(x_n'-x_n'')^2\}}$.

Now we have points, straight lines, and distance. I suppose that *angle* is the next thing to define. But before we do that I would like to mention the *triangle inequqlity*.

The propositions of Euclid occupy a unique position in mathematics. When we are at school we know all about them, and when we leave school we forget all about them. It is surprising how much mathematics you can do in this state of forgetfulness. Euclid is like a world apart. A few years ago I wanted to make sure that the perpendiculars of a spherical triangle met at a point. I feel that they must, but the proof seemed rather difficult, and I could not find it in a book. So I thought that perhaps one could in some way generalise or work from the proof of this well-known theorem for a plane triangle. I found that I had completely forgotten Euclid's proof. So I tried to reconstruct the proof and failed. I had to look it up. The only proofs I could make up were analytical and clumsy. So I got a new respect for Euclid.

Now hidden among Euclid's theorems is one that states that the sum of two sides of a triangle is greater than the third side. It is one of those rather obvious things that are boring to prove. But that theorem or result under

the name of the *triangle inequality* plays an important part in general geometrical theories.

So at once we ask : Is the triangle inequality true in four or n dimensions?

You will realise that we are asking a question with a meaning, for we have defined points and the distances between them. Let us write down the triangle inequality in n dimensions and see what it looks like. For simplicity let us take one of the three vertices of the triangle at the origin O ; let the other two vertices be $P'(x_1', x_2', \ldots x_n')$ and $P''(x_1'', x_2'', \ldots x_n'')$. One of the three possible inequalities for the triangle $OP'P''$ reads :

$$OP' + OP'' > P'P''.$$

Translating this into symbols in terms of our definition of distance, the triangle inequality reads :

$$\sqrt{(x_1'^2 + x_2'^2 + \ldots + x_n'^2)} + \sqrt{(x_1''^2 + x_2''^2 + \ldots + x_n''^2)} > \sqrt{\{(x_1' - x_1'')^2 + (x_2' - x_2'')^2 + \ldots + (x_n' - x_n'')^2\}}.$$

To prove the triangle inequality we have to prove this algebraic inequality.

It looks as if we were in for some tedious algebraic manipulation at this point. That is, if we can get started. But how are we to *start* to prove this inequality?

It may be that to you the method of proof is obvious. But in case there is someone present to whom it is not obvious, I would like to give him or her a little help.

In the first place, this is a theorem about a triangle. A triangle, even if it happens to lie in a space of n dimensions, is itself only a two-dimensional figure—the triangle that every schoolboy is familiar with. Can we use that fact?

Yes, we can. We can change our coordinate system. In three-dimensional space we are accustomed to the process of rotating the coordinate system. Although we may write down the mathematical equations corresponding to such an orthogonal transformation, our basic appreciation of a rotation of axes is intuitional—we *see* the rotation taking place. There is no harm in that. But if we wish to carry over into n dimensions the idea of a rotation of coordinate axes, we must leave the intuition behind and carry over only the formulae, suitably generalised so as to apply to n dimensions instead of only three.

I do not want to bore you by writing down the formulae for an orthogonal transformation in n dimensions. I shall ask you to believe that it is possible to change the coordinate system in such a way that in the new system we have the following coordinates for our three points :

$$\begin{aligned} O &: \quad (0, 0, \ldots 0). \\ P' &: \quad (x_1', x_2', 0, 0, \ldots 0), \\ P'' &: \quad (x_1'', x_2'', 0, 0, \ldots 0). \end{aligned}$$

Then the triangle inequality takes the simpler form :

$$\sqrt{(x_1'^2 + x_2'^2)} + \sqrt{(x_1''^2 + x_2''^2)} > \sqrt{\{(x_1' - x_1'')^2 + (x_2' - x_2'')^2\}}.$$

Can you prove that inequality to be true no matter what values may be given to the four x's involved?

If you cannot prove it, the situation is indeed tragic. You are confessing that you cannot prove an elementary proposition of Euclid—the proposition that two sides of a triangle are together greater than the third.

In desperation, you may reason as follows. First, you look up Euclid's proposition and go through the proof. That, you say, is a good proof. But

the above inequality is equivalent to Euclid's proposition. Therefore the above inequality is true.

But that is, in a sense, cheating. It may get you past this difficulty, but you will run up against it later in another form, and you will not be able to extricate yourself like this. We should prove the inequality algebraically, without recourse to the proposition of Euclid.

The Schwarz inequality.

The proof is really not hard. We note that each side of the inequality is positive, and so it is equivalent to the inequality obtained by squaring both sides. When we square, certain terms cancel out, and we are left with :

$$\sqrt{(x_1'^2 + x_2'^2)} \cdot \sqrt{(x_1''^2 + x_2''^2)} > - x_1'x_1'' - x_2'x_2''.$$

We have divided across by 2, an operation which does not change an inequality.

Please remember that the inequality last written has *not* been proved ; we are trying to prove it. Let us lay it aside for a moment, and write down something that is certainly true :

$$(px_1' + x_1'')^2 + (px_2' + x_2'')^2 \geqslant 0.$$

No matter what (real) values we give to the five numbers here involved—p and the four x's—this inequality is certainly true. But this is the same as

$$p^2(x_1'^2 + x_2'^2) + 2p(x_1'x_1'' + x_2'x_2'') + (x_1''^2 + x_2''^2) \geqslant 0.$$

Now here we have a quadratic expression in p, which might be written

$$ap^2 + 2bp + c,$$

and we know that it cannot be negative for any value of p. The quadratic equation

$$ap^2 + 2bp + c = 0$$

cannot have distinct real roots. By elementary algebra, then, we know that the discriminant is negative or zero. Thus

$$b^2 - ac \leqslant 0,$$

or

$$(x_1'x_1'' + x_2'x_2'')^2 \leqslant (x_1'^2 + x_2'^2)(x_1''^2 + x_2''^2).$$

From this, the triangle inequality follows at once. The equality sign corresponds to a flat triangle. I shall not delay over that.

It is really just as easy to prove the triangle inequality with all the $2n$ coordinates—it is unnecessary to transform so that only four coordinates remain. The key is the obvious inequality :

$$(px_1' + x_1'')^2 + (px_2' + x_2'')^2 + \ldots + (px_n' + x_n'')^2 \geqslant 0.$$

I have spent some time on the triangle inequality, because I think it deserves it. It is one of the strongest links between the intuitional geometry we are all familiar with and the more abstract analysis of many-dimensional spaces. The key to this connection is the *positive definite form*, and we may give the general name *Schwarz inequalities* to the type of inequality we have been discussing.

Actually, the name Schwarz inequality is usually reserved for a certain inequality involving integrals. That inequality also has a geometrical significance, and I shall write it down.

Let $f(x)$ and $g(x)$ be any two functions and p any constant. Then it is evident that

$$\int_0^1 (pf(x)+g(x))^2 dx \geqslant 0,$$

because the integrand is not negative anywhere. Hence, on expanding the square, we have

$$ap^2+2bp+c \geqslant 0,$$

where a, b, c are certain integrals. We conclude that

$$b^2 \leqslant ac$$

or

$$\left(\int_0^1 fg\,dx\right)^2 \leqslant \int_0^1 f^2\,dx \,.\, \int_0^1 g^2 dx.$$

This is the usual Schwarz inequality. It is the key to the study of function space.

The mention of function space reminds me that I am half-way through this lecture, and that I must reserve time to talk about function space. But I am not yet ready to leave the space of four or n dimensions.

Angles.

I was going to define *angle*, and then I got deflected into the triangle inequality.

In ordinary geometry angle is an intuition. In four or more dimensions it must be analytically defined. One plan is to use a familiar formula of trigonometry :

$$\cos C = (a^2+b^2-c^2)/2ab.$$

When we meet that formula in trigonometry, we are asked to prove it. The situation is now reversed. This formula is the *definition* of the angle C in terms of the lengths of the sides of the triangle, length having already been defined.

When we approach the formula from this side, an interesting possibility presents itself. Suppose that when you insert the lengths of the sides of the triangle, the quotient on the right comes out to have the value 3. You then search your trigonometric tables for an angle having the cosine 3. You won't find it, because no angle has a cosine less than -1 or greater than $+1$. In such a case, then, the formula fails to define an angle.

However, this cannot occur—at least not in the geometry we are considering. Cos C, as defined above, cannot exceed unity in absolute value. This is easy to see. We have

$$\begin{aligned} 1-\cos C &= 1-\frac{a^2+b^2-c^2}{2ab} \\ &= \frac{c^2-(a-b)^2}{2ab} \\ &= \frac{(c+a-b)(c-a+b)}{2ab}. \end{aligned}$$

The triangle inequality assures us that this cannot be negative, and so cos C cannot exceed 1. In the same way we show it cannot be less than -1. Thus our formula always defines a real angle in the range $0 \leqslant C \leqslant \pi$.

The four-dimensional box.

It would be a pity if, in four or more dimensions, our geometry got lost in algebraic formulae. Near the beginning of this lecture, I talked about a box in four dimensions. I would like now to draw a picture of such a box.

If you ask me to draw a sketch of an ordinary box, I draw something like this (Fig. 1) :

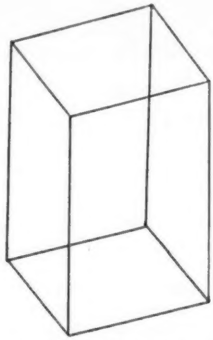


FIG. 1. Three-dimensional box.

This is, of course, a two-dimensional projection of a three-dimensional box, and you probably regard it as a fairly satisfactory picture. In the same way, I hope that you will be satisfied with a two-dimensional projection of a four-dimensional box.

Perhaps you don't know how to draw a picture of a three-dimensional box. The best plan is to draw two parallelograms with edges equal and parallel. Then you join the corresponding corners.

The same plan works in four dimensions.

I draw two three-dimensional boxes, with corresponding sides equal and parallel. Then I join the corresponding corners (Fig. 2). At each corner I

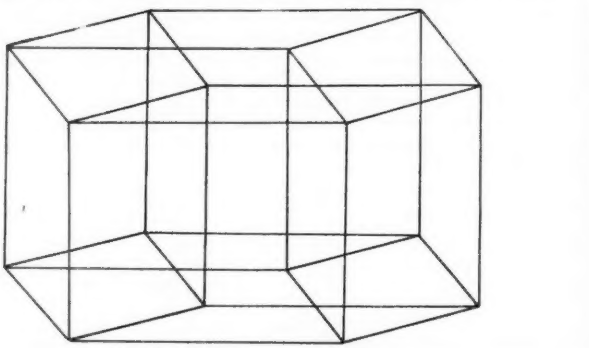


FIG. 2. Four-dimensional box.

have four mutually perpendicular edges. Of course, you don't expect them to *look* perpendicular, on account of the perspective. It is a good plan to draw with different colours, using a single colour for all edges that are parallel to one another.

If you want to see a five-dimensional box, you have merely to draw two of

these four-dimensional boxes with corresponding edges equal and parallel, and then join the corresponding corners.

But let us not wander so far. You see the corners of the four-dimensional box. You can count them ; there are 16 *corners*, just twice as many as the corners of the three-dimensional box. You can also count the edges. There are the edges of the two three-dimensional boxes, and then there is an edge for each corner of one of these boxes. That is to say, $2 \times 12 + 8 = 32$ *edges*.

What about faces? A four-dimensional box has two sorts of faces—two-dimensional faces and three-dimensional faces. It looks as if we needed a new word. But the best plan is to pick out one of the words we use for an ordinary box—say, the word *face*, and use it for everything. Thus a corner is an 0-face, an edge a 1-face, and then we shall have 2-faces and 3-faces. This language has the advantage that it can be carried right on into any number of dimensions, and we do not have to invent a new word every time we increase the dimensionality by one.

How many 2-faces has a four-dimensional box? Each 2-face appears as a parallelogram in our figure, and we can count them if we want to. But it is easier to see that we have all the 2-faces of the two 3-boxes we have used (that is, $2 \times 6 = 12$), and in addition each edge of one of the 3-boxes generates a 2-face. So the total number of 2-faces is $2 \times 6 + 12 = 24$.

Finally, what about 3-faces? Each 3-face appears in our figure as a parallelepiped. We have the two original 3-boxes, and then each 2-face of one of these 3-boxes generates a 3-face. So we have altogether the following number of 3-faces : $2 + 6 = 8$.

We may sum up as follows :

	2	3	4	5	n
0-faces - - -	4	8	16		
1-faces - - -	4	12	32		
2-faces - - -	1	6	24		
3-faces - - -		1	8		
4-faces - - -			1		
...					

I leave the rest of this interesting numerical table for you to complete.

Spheres.

In four dimensions a sphere is easy to define. The sphere with centre at the origin and radius unity has the equation

$$x^2 + y^2 + z^2 + w^2 = 1.$$

That is to say, the sphere consists of all those points for which the sum of the squares of the four coordinates is unity. If you ask me to draw a picture of it, the best I can do is to draw a circle.

Does a sphere divide our four-dimensional space into two parts, an interior and an exterior? Does an ordinary sphere in our ordinary space divide it into two parts? Any child can answer that. When a child bursts his rubber ball, his grief is tempered by joy at the realisation that at last he can see what it is like inside. But unless he bursts the ball, he cannot get inside.

So, if we lived in a four-dimensional world and had four-dimensional intuitions, you would rightly regard me as a hair-splitter for asking the question at all. But we have not got those intuitions and must go slowly.

The question is this : Can we make up a simple definition of *interior* and *exterior* such that the interior and the exterior are accessible to one another only by passing through the sphere?

The definitions are very simple. Consider the expression

$$x^2 + y^2 + z^2 + w^2 - 1.$$

You can calculate its value at any point of our four-dimensional space by simply sticking in the values of the four coordinates of that point. The value is a number, which must, of course, be negative, zero, or positive. If the value is *negative*, we say that the point lies in the *interior* of the sphere, if it is *zero* the point lies (we know) *on* the sphere, and if the value is *positive*, we say that the point lies in the *exterior* of the sphere.

To pass continuously from the interior to the exterior, we must change the value of this expression continuously from negative to positive, and we can do this only by making it pass through the value zero. But when the value is zero, we are on the sphere. So the interior and the exterior are accessible to one another only by passing through the sphere.

This means that a four-dimensional child, playing with his four-dimensional ball, would have to burst it if he wanted to see what the inside was like. But his parents could buy him a three-dimensional ball, and that he could explore completely without bursting it, just as a three-dimensional parent could buy a destructive three-dimensional child a hoop to play with. But, indeed, the four-dimensional child would have a greater variety, for his parents, if they could afford it, might provide him with a four-dimensional ball, a three-dimensional ball, and a two-dimensional hoop. The variety in an n-dimensional toy-shop (n greater than 4) would be even greater.

Connectedness.

But there are other questions to be asked about the sphere in four dimensions. Is it connected? You know that you cannot go to the moon without leaving the surface of the earth, but you can go to New York without leaving the surface of the earth, advertisements of airlines notwithstanding. In fact, the surface of the system earth + moon is not connected; the surface of the earth is connected.

To raise the question of the connectedness of an ordinary sphere is to court ridicule. You have only to look at a sphere to *see* that it is connected. But we cannot *look* at our four-dimensional sphere. What assurance have we that it is actually connected?

But before we ask the question, we must know what we mean by "connectedness". It is very simple. We have two given points on the sphere,

$$P' \text{ with coordinates } x',\ y',\ z',\ w',$$

and

$$P'' \text{ with coordinates } x'',\ y'',\ z'',\ w''.$$

When we say that these points lie on the sphere, we mean that these eight numbers satisfy the two equations

$$x'^2 + y'^2 + z'^2 + w'^2 = 1,$$
$$x''^2 + y''^2 + z''^2 + w''^2 = 1.$$

The question is this: Can we continuously transform P' into P'' without leaving the sphere? Mathematically expressed, can we find four continuous functions of a parameter t, say,

$$x(t),\ y(t),\ z(t),\ w(t),$$

such that for $t=0$ these four functions take the values

$$x',\ y',\ z',\ w' \text{ respectively,}$$

and for $t=1$ the values

$$x'',\ y'',\ z'',\ w'' \text{ respectively,}$$

with a further important condition, namely, that for every value of t in the range $(0, 1)$ the sum of the squares of the four functions shall be unity?

Such functions can be found. The sphere is, in fact, connected. I shall

not insult your intelligence by filling in the details. I shall make just one remark of a philosophical nature. Most of us prefer to be slaves rather than free men. We find it easier to solve a problem which admits a unique solution (or perhaps a finite number of solutions), rather than one in which the number of possible solutions is infinite. It is clear if there is one path from P' to P'', there must be an infinite number, just as there are indefinitely many ways to go from Dublin to Cork. If you ask a man for the shortest route from Dublin to Cork, he will be delighted to help you. But if you merely ask for *a* route, he will be unhappy, wondering whether you really want to go via Belfast or Ballina. It is just the same in mathematics. Some slavish instinct in us makes us prefer to work by rule rather than exercise our imaginations. And the plan we usually adopt, in the case of a problem which embarrasses us by the freedom it offers, is to impose voluntarily certain restrictions which seem to make for simplicity. In our problem of the sphere, the tip I would give you is this : To go from Dublin to Cork, first go straight north from Dublin to the North Pole, and then due South from the North Pole to Cork.

Ellipsoids and hyperboloids.

Time is passing, and I have not got to function space yet. Just a few words more about n-dimensional space.

What about our old friend the ellipsoid? Does it pass over easily into n dimensions? Yes, very easily, by the equation

$$x_1^2/a_1^2 + x_2^2/a_2^2 + \dots + x_n^2/a_n^2 = 1.$$

The ellipsoid is connected, and it has an interior and an exterior.

What about hyperboloids? In ordinary space, all ellipsoids are very much alike. Some are long and thin and some are short and fat, but they are all recognisable as members of one family. But in the case of hyperboloids, there are, as we well know, two quite different types. There are hyperboloids of one sheet and hyperboloids of two sheets, and no one could possibly mistake the one for the other. The hyperboloid of one sheet is a connected surface while the hyperboloid of two sheets consists of two parts.

This raises some interesting questions. In n dimensions, how many types of hyperboloids are there? How many of these are connected? And in the case of the unconnected ones, into how many parts are they separated?

I shall answer these questions briefly. You get hyperboloids by throwing minus signs into the equation of an ellipsoid. Obviously, you cannot make all the signs minus, and so in n dimensions there are $(n-1)$ types of hyperboloid. (Check : for $n=3$, we get $3-1=2$ types.) The question of connectivity is not dealt with so quickly, and I shall merely state the result. Of the $(n-1)$ types, all are connected except one type. That one unconnected hyperboloid has just two parts, like our familiar hyperboloid of two sheets.

At last I am ready to go on to function space. But just one interpolation. What about a space with -3 dimensions, or with π dimensions? The idea is not entirely meaningless, but I do not think we should pursue it further here.

What I want to get on to is the space with an infinite number of dimensions.

Function space.

A hundred years ago no respectable mathematician cared to be caught talking about space of dimensionality higher than three. And a space of infinite dimensionality was something worse still. Now that the conventions have been broken, we may have gone to the other extreme ; we may be in danger of debauching geometry by driving intuition out of it too completely.

One way of discussing space of infinite dimensionality is to work in n dimensions and then let n tend to infinity. What I have to say might be

introduced along those lines, but I think that it is better to make a clean break and start in a clear if artificial way.

Consider a real variable x, which can take values from 0 to 1. Consider now some functions of x, say

$$x^2, \quad 3x^2, \quad x^3, \quad 3x^2+x^3.$$

We notice certain relationships between these functions. Thus, the second function is obtained from the first by a very simple process, namely, multiplication by a constant. The last function is formed from the second and third by addition.

It would undoubtedly be helpful to have some sort of chart of functions. It is true that we can draw a graph of each function, but graphs do not show up the kind of relationships we have just been talking about. So we try another plan.

Take a sheet of paper. On this sheet of paper we are going to represent each function by a dot or point. At first we have no guide except a general rule—to each point there should correspond only *one* function.

On the sheet of paper I mark then four points at random (Fig. 3). Together with these four points I have inserted another, corresponding to the simplest function of all—the function zero.

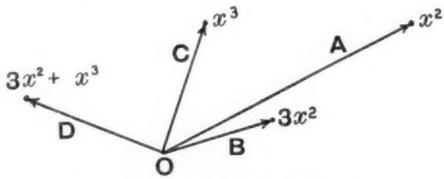


FIG. 3. Representation of functions.

Instead of thinking of the functions as points, we can think of them as *vectors* drawn from the point O. We can give those vectors names like this:

A corresponds to x^2,
B corresponds to $3x^2$,
C corresponds to x^3,
D corresponds to $3x^2+x^3$.

We might add the zero vector:

O corresponds to 0.

Now we are all familiar with the parallelogram law for vector addition. Using this law, we can construct on our paper the vector $B+C$. To what function should we make this vector $B+C$ correspond? Would it not be natural to make it correspond to the sum of the functions which correspond to B and to C? That would mean:

$$B+C \text{ corresponds to } 3x^2+x^3.$$

But the vector D already corresponds to this function. Therefore our vectors should not be drawn quite at random on the paper. We should draw them so that

$$B+C=D.$$

Further, from the relationship between the functions x^2 and $3x^2$, it suggests itself naturally that the vector B should be drawn in the same direction as A and three times as long.

Thus, instead of the original chaotic diagram, we now have this one, which displays graphically the relationships between our functions (Fig. 4).

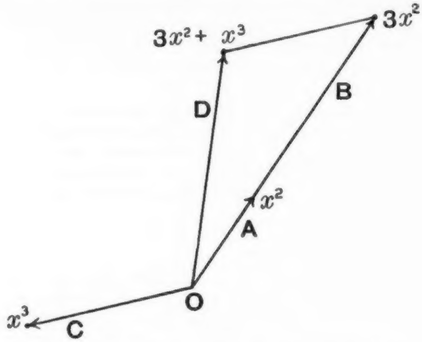


FIG. 4. Revised representation of functions.

Infinite dimensionality.

It looks then as if there is profit in this graphical representation of functions by vectors in a plane. But difficulties lie ahead.

Consider the functions x^2 and x^3, with the corresponding vectors A and C. Then consider the function

$$ax^2 + bx^3,$$

where a and b are any two constants; by our rules it corresponds to the vector

$$aA + bC.$$

If we give to the constants a and b all possible values, the extremity of the vector $aA + bC$ takes up all possible positions on the plane. Does that mean that every function of x can be represented in the form

$$ax^2 + bx^3?$$

Certainly not. The function x^4, for example, cannot be so represented. It looks then as if our method of representation is dangerous, and may easily lead us to make false conclusions.

But we can rescue it. Suppose we regard our sketch merely as a projection on the paper of a vectorial scheme in multi-dimensional space? Then it may be true that functions of the form

$$ax^2 + bx^3$$

exhaust a two-dimensional plane in our multi-dimensional space, but the vector representing a function like x^4 may be taken care of by a vector which lies outside that plane.

This works, but at a price. On account of the fact that there are infinitely many functions to take care of, no n-dimensional space will suffice. We must consider our vector space to possess *infinitely many dimensions*.

This is the function space I wanted to talk about, or at least a simple example of it. A point or vector in function space might correspond to a set of functions of several variables. Let us stick to the simplest case—one function of one variable.

To sum up, then, our basic laws are as follows :

If vector	A corresponds to function $f(x)$
and vector	B corresponds to function $g(x)$,
then	$A+B$ corresponds to $f(x)+g(x)$
and	kA corresponds to $kf(x)$,

k being any constant.

You will remember that in setting up our dictionary for a space of four or n dimensions, we first defined a point and then a straight line determined by two points. In function space we have defined the point (it is merely a variation of language to call it a vector). The straight line determined by $f(x)$ and $g(x)$ is easy to define : it is the set of functions of the form

$$af(x)+bg(x) \quad (a+b=1).$$

Distance or length in function space.

The next thing to define is *distance*, and here the fun begins. In order not to confuse the issue with too many ideas at once, I did not go into the question of the various definitions of distance which might be employed in space of a finite number of dimensions. A simple definition suggested itself, and I took it. But in function space, what are we to do? How are we to define the *length* of the vector which corresponds to a function $f(x)$?

This is a crucial point. If no man comes forward with a definition, we are stuck with a space of infinite dimensionality, but without the concept of distance in it. On the other hand, someone may offer a definition which leads to a complicated and uninteresting geometry. Happily we have a definition —Hilbert's definition—which works like magic. It is not by any means the only definition, but I shall not talk about any other.

Following Hilbert, we define the length of the vector corresponding to $f(x)$ to be *

$$\sqrt{\left\{\int_0^1 (f(x))^2 dx\right\}}.$$

Certain pleasant features of this definition are obvious. In the first place, we shall not be led into complications with imaginary lengths—the quantity of which we have to take the square root cannot be negative, since the integrand is a square. In the second place, there is only one vector of zero length, and that is the vector which corresponds to the function zero.

But there is more in the definition than that. What about the triangle inequality? Suppose we consider the triangle formed by the origin (zero function) and the points corresponding to the functions $f(x)$ and $g(x)$. One of the triangle inequalities for this triangle reads :

$$\sqrt{\left\{\int_0^1 (f(x))^2 dx\right\}}+\sqrt{\left\{\int_0^1 (g(x))^2 dx\right\}}>\sqrt{\left\{\int_0^1 (f(x)-g(x))^2 dx\right\}}.$$

Is that true, for any two functions of f and g? Yes; it is easily proved by means of the Schwarz inequality. The triangle inequality is true in our function space.

Angle is easily defined, and the triangle inequality ensures the reality of angles. In fact, the development of the geometry of function space is almost monotonously easy. It is true that there is one feature to which we do not

* This definition involves a range of integration. For simplicity, we have taken it to be (0, 1); we might more generally take it to be (a, b). The only requirement is that the integral shall exist; for example, we could not use the range $(-\infty, \infty)$ if we wanted to define the length of the vector corresponding to the function x^2.

become readily accustomed; at any one point we can draw an infinite number of mutually perpendicular lines. But as long as we stick to a finite number of directions, function space is just like the n-dimensional space we have been discussing earlier. Indeed, we realise with a shock that it is not merely just like—it is the same.

Let me bring out this point. Suppose we take just two functions, say, x^2 and x^3, and do not let our minds wander outside the set of functions $ax^2 + bx^3$, where a and b are constants taking all values from minus infinity to plus infinity. That means that we are sticking to one plane in function space. Then I assert that the geometry of that plane is absolutely identical in every respect with the familiar Euclidean geometry of the plane. The only difference lies in the interpretations of words such as point and straight line in terms of ideas which lie outside the geometry proper.

The real and the complex.

That brings me to the end of what I have to say. But I shall add one remark about imaginary and complex coordinates. Every student of plane geometry is thrilled to learn that a circle passes through two imaginary points at infinity. It is an indecent and disloyal thrill of which he should be ashamed. The geometry of the plane with points having complex coordinates is not the geometry of two dimensions but of four, and it is only the bullying algebraist who holds the contrary. At the other end of the scale, the quantum theorist insists on having everything complex in his function space. What he succeeds in doing with this complex function space is indeed marvellous, but I hold that the concept of function space has in itself nothing to do with complex numbers, and that it should be explored, at first at least, in terms of real elements only. As a last word, let me remark that the whole vast theory of functions of a complex variable could never have been constructed without the Argand diagram, in which the point has two real coordinates. Complex numbers, treated as entities and not as number-pairs, are a pain in the neck to the true geometer. J. L. S.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

Dear Sir,—In his review of my *Theory and Application of Mathieu Functions*, Mr. T. V. Davies says: "The reader, however, who expects the miscellaneous integrals to be of the same comprehensive and complex variety found in Watson's *Bessel Functions*, will be disappointed with Chap. XIV." I share the reviewer's disappointment, but am unaware that such integrals are extant. I contributed some 40 per cent. of new material in the text, and I think someone else might supply the missing integrals.

It would not have been difficult to increase the length of the book by 50 per cent. using existing material. But in these miserable days of almost astronomical printing costs and evanescent paper supply, an author must perforce be eclectic rather than exhaustive.

I take this opportunity of correcting some errors:

p. 17, in (6), for 1109 read 609.

in (8), (9), for 17 28000 read 27216 00000.

p. 310, l. 2, for k_1^2 read k_1^4.

l. below (6), for $\omega h/c$ read $(\omega/c)^{\frac{1}{2}}h$,

and for $\omega h^2/4c^2$ read $\omega h^2/4c$.

p. 313, in (1) and in the third line above (5), for ω/c read $(\omega/c)^{\frac{1}{2}}$.

N. W. McLachlan.

THE NATURE OF MAIN-SCHOOL GEOMETRY.*

BY C. V. DURELL.

THERE are in every kind of school a very large number of pupils of average intelligence who have no special aptitude for mathematics. The range of work represented by the alternative syllabuses may be regarded as the upper bound of the mathematical studies of these pupils. The suggestions which I shall make refer primarily to such pupils, but I should like them to apply to all who have not exceptional mathematical ability.

There can be no doubt that, whatever form examinations take in the next ten years, the movement which led to the construction of the alternative syllabuses will exercise an increasing influence on the character of main-school mathematics, and especially geometry. This movement has not sprung into existence all of a sudden ; it is the natural outcome of the revolutionary changes set on foot more than fifty years ago, which Lewis Carroll called a struggle between Euclid and his modern rivals.

Euclid's *Elements* is a textbook of axiomatic (or abstract) geometry. Its points are undefined entities and its axioms are statements about undefined relations between these entities. It is not a geometry of physical space. All the defects modern scrutiny has detected in Euclid's proofs are due to conscious or unconscious appeals to spatial experience, for example, his use of superposition.

The fact of the matter is that Euclid's geometry is far easier for disembodied spirits than for us humans. We are heavily handicapped by preconceptions arising from our sense of physical space which make us liable to say all sorts of things without realising they need justification. On the other hand, disembodied spirits are not merely handicapped, but feel fog-bound, when they try to study Stage B geometry ; and it appears to me important we should recognise this fact.

Take what is probably the first theorem we profess to prove in Stage B, the exterior angle property :

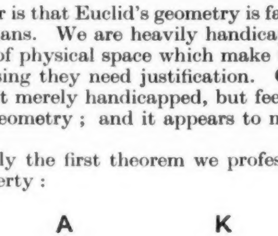


FIG. 1.

We draw CK parallel to BA so that K and A are on the *same side* of BC; we do not hesitate to assume that a line has two sides. We then continue without a twinge of conscience to assume that K and B are on opposite sides of AC, although the remainder of our proof is worthless if this is not true.

What does the unhappy disembodied spirit think about all this? He will ask, Why do the axioms show that a line has two sides, and why are K and B on opposite sides of AC? These are questions which he approaches with a completely open mind ; he is just as uncertain about the answers as my

* A paper and discussion at the Annual Meeting, Birmingham, 1949.

housekeeper when she tried to decide whether to put back or put forward the hands of her alarm clock on the night before summer time started. Each has no idea of what the answer will be until the problem has been settled by reasoning. A disembodied spirit with inadequate intelligence and a housekeeper with hazy ideas about Greenwich time may both fail to reach the correct conclusion.

I asked Mr. Robson yesterday what he would do about this, and probably all of you will agree with his reply ; he said : " In Stage B geometry I am not interested in sides of a line, and I start from any convenient group of assumptions."

I support unreservedly the practice of starting from a broad basis of assumptions, but when the list has been drawn up, we should confine ourselves to it. Amiable disembodied spirits will accept any consistent set of assumptions we propose, but no set, suitable for school use, has been suggested which will dispel the fog surrounding our proof of this first Stage B theorem. Our set of assumptions must be in fact all those properties of physical space which we regard as obvious ; this is indeed the essence of Stage A geometry. All the elements of this geometry are elements of physical space, and there is no fairy's wand which can suddenly transform them into the undefined entities and undefined relations of a spirit world.

The somewhat imperfect system of axiomatic geometry as represented by Todhunter's edition of Euclid's *Elements*, which was in use at my first school, made most pupils regard geometry as mumbo-jumbo. Enunciations of theorems meant as little as the proofs which followed. The text of the sermon which the revolutionaries were preaching fifty years ago amounts to this :

Schoolboys are not disembodied spirits and must not be treated as though they are.

This is the reason why the first object of the reforms was to help pupils to understand statements about lengths of lines and sizes of angles by basing the work on the use of drawing and measuring instruments.

In this approach, a straight line is the name given to what is drawn by a pencil and ruler. From the point of view of the pupil, this drawing *is* a line. Also the length of a line AB is a quantity obtained by using a graduated ruler. When the pupil says $AB = CD$, he means that the ruler-measurements of AB and CD are the same, without worrying about what a length of, say, 1 inch is. Similarly, the meaning he attaches to a statement about the size of an angle is derived from the use of a protractor. This is the procedure of Stage A geometry ; its elements are physical objects and replace the undefined entities and undefined relations of Greek geometry which Euclid tried to systematise.

The revolutionaries succeeded in abolishing Euclid's *Elements* as a school textbook, but all changes must be gradual—there are limits to the shocks humanity can bear—and so, as regards Stage B, the books which were substituted contained much, both as to matter and method, which was included only because it formed part of the *Elements*, and to a lesser extent this is true of all books in general use to-day. Before justifying this remark, it will help to recall Euclid's purpose : he attempted to start from a small group of assumptions and forge a logical chain, proceeding from one exact statement to another and ending with the construction of the regular solids. It was essential for his purpose to construct a theory of incommensurables, and until this had been done, ratios were inadmissible. Further, it is important to notice that Euclid, while saying this line equals that line or this figure equals that figure, never says that the *length* of this line equals the *length* of that line or the *area* of this figure equals the *area* of that figure ; the words, length,

area, etc., as used in the main school have no place in Euclid's geometry. Perhaps this distinction can be indicated by saying that Euclid's geometry is metrical but is not mensurational.

Everyone, looking at the alternative syllabuses for the first time, must be deeply impressed by the increase of range proposed. The inevitable question then follows: Has enough been cut out of the ordinary syllabus to make room for the new material? I think it has, if we regard the remodelled syllabus as the natural fulfilment of the changes started fifty years ago. We are neither able nor anxious to retain Euclid's continuous logical chain of theorems. For a long time past, the list of theorems in the Stage B course has been shortened, and recently it has been proposed that attention should be concentrated on about a dozen key-theorems. In the alternative syllabuses we now find few or no theorems, although some questions, in which short and simple geometrical reasoning is required, are retained. Most teachers will agree that such reasoning is a valuable part of main-school geometry, whether applied to numerical or general data.

Now if we look at the consequences of Euclid's self-imposed conditions, the most striking fact is that it was necessary for congruence to assume the leading role. It seems certain that Euclid's influence is responsible for the prominence of congruence in present-day school geometry. It has taken fifty years to raise the status of similarity up to the level of congruence, and I suggest that the time has now come to relegate congruence to a subordinate position. It takes time to accustom the pupil to the meaning and use of a ratio, but the value of such work is unquestioned; and I suggest it is often preferable in geometry to use ratios where hitherto we have used equalities. For example, the group of intercept properties is usually established by congruence, working with equal intercepts. Is it not preferable to work with ratios of intercepts, following the same line of argument, but replacing congruent triangles by similar triangles? Thus the mid-point theorem appears merely as one special case of a general property of which, for example, practical use is made in the diagonal scale.

I suggest further that pupils gain little or nothing from arguments based on congruence, when employed to prove those equalities associated with the parallelogram, rectangle, square, rhombus and isosceles triangle, which they regard as obvious. Such arguments do not make the pupil feel any more certain about the truth of the statement. Is it not also reasonable to maintain that the value, which this particular technique of presentation possesses, is secured by using similarity tests in place of congruence tests?

If proofs are not required for equalities which appear to be obvious, the need for using congruence tests is rarely felt. But when the pupil uses arguments based on the similarity tests, he establishes results which certainly do not appear obvious, and he is learning to use methods of wide application and practical importance. For example, pupils use trigonometrical methods more intelligently if in the initial stages more stress is laid on the actual ratio than on the trigonometrical notation for the ratio.

I ask next whether establishing equality of area by what may be called the "same base and between the same parallels" method is worth retention. Is not the use of this method merely due to the fact that Euclid employed it for reasons with which we are not concerned? If area is a quantity-measurement, is it not more natural to proceed in turn from the rectangle to the right-angled triangle and then to any triangle by subdivision into right-angled triangles, expressing the results as mensuration formulae and then making use of such formulae as occasion may require? Is the area of a parallelogram really a matter of any importance? How often do we need to make any use of it? Is not its prominence in Stage B geometry another

instance of Euclid's influence, and may we not now regard it as merely a decorative feature of the background of our geometrical picture? I have in my study a china dog; I like it because it used to stand on the mantelpiece in my nursery, and so I pay no attention to complaints that it collects the dust. Replace my china dog by the construction for reducing a quadrilateral to an equivalent triangle. Is our affection for this particular construction due to the fact that we have known it for such a long time? It dates from Simson's edition of Euclid, 1780. Is it possible that since then it has collected some dust?

As regards Pythagoras' theorem, I agree with those who regard it primarily as a theorem about area and only secondarily an algebraic relation, although the latter form is the more useful. But I am bound to admit that I do not feel very confident I could stand up to a severe cross-examination. Still I do think that the merit of Euclid's proof is its emphasis on the area-aspect. This can, however, be retained while replacing his use of congruence and of the "same base and between the same parallels" method by trigonometry.

If, as is now customary, the theorem is approached via an acute-angled triangle, we can say with the usual notation:

$$AF = b \cos A, \quad \therefore \text{ area 1 is } c \,.\, b \cos A\,;$$

$$AE = c \cos A, \quad \therefore \text{ area I is } b \,.\, c \cos A\,;$$

and so $\quad$ area $1 =$ area I $= bc \cos A$.

Similarly, $\quad$ area $2 =$ area II $= ac \cos B$,

and $\quad$ area $3 =$ area III $= ab \cos C$.

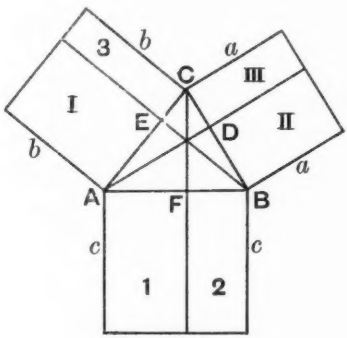


FIG. 2.

This shows vividly in terms of areas why

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

and the other two corresponding results. In particular when $C = 90°$, we have $c^2 = a^2 + b^2$.

Differences of opinion about details are inevitable and perhaps are unimportant. The reason why I have entered into details is to illustrate what appears to me a matter of fundamental significance. It is this:

The alternative syllabuses have replaced Stage B geometry by an advanced Stage A course in which admittedly all elements are elements of physical space.

For example, we use arguments which take a form such as:

$$\text{if} \quad PQ = b \text{ in.}, \quad PR = c \text{ in.}, \quad \angle QPR = x^\circ,$$

then area $\triangle PQR = \frac{1}{2}bc \sin x^\circ$ sq. in.; hence, etc.

The introduction of plan and elevation is another instance of advanced Stage A work. The emphasis is on practical geometry. A discussion at the Association's Annual Meeting in 1946 showed that many teachers felt such work could be studied with profit in the Sixth Form.

We may feel it is worth while to encourage pupils to draw lines of grade α rather than grade ω, but I suggest that whether we do so or not, we should in the main school allow pupils to regard what they have drawn as the line itself which is the subject of discussion and steer clear of the undefined entities and relations of Greek geometry. All lines in the main school have some breadth, and the length of a line is expressed in terms of some unit, inches, or cm., to as high a degree of accuracy as the occasion requires. The pupil takes this as a matter of course because his natural inclination is to think of the geometry he studies as the geometry of physical space.

May I put to you three questions about the sentence:

the length of $AB =$ the length of CD.

(i) What do *you* mean when you say this?
(ii) What does *the pupil* mean?
(iii) What do *you* wish *the pupil* to mean?

If you agree that schoolboys should not be treated like disembodied spirits, it appears to me that the answers to the last two questions must be that they are statements about material objects. There should be no market for disembodied rulers and disembodied protractors.

It is true, of course, that in the geometry of physical space, as in every branch of physics, all statements about quantities must be approximate. There is no such thing as a length of *exactly* one inch or an angle of *exactly* 60°. But approximations are a part of everyday life; they should not be regarded as blemishes like the common cold.

Mr. **Robson** agreed with a great deal of what Mr. Durell had said. He expressed surprise that the alternative syllabuses had not yet been universally accepted. To find time for new work, economies should be made in formal geometry, especially with standard theorems. There should be a liberal basis of assumptions, so that pupils did not spend their time writing out proofs (*e.g.* by congruence) of results which appeared to them to be obvious. Similarity should be taken early; and area should have a mensurational bias, so that, for example, the use of a formula $\frac{1}{2}bh$ should be preferred to results about triangles between the same parallels.

He did not, however, regard congruence as a mere special case of similarity, but rather as a particularly important case which should be taken first: some congruence results are not trivial. In elementary teaching, easy special cases should often be taken before the general case; and for this reason it may be better to take the theorem of Pythagoras before the general cosine formula. He questioned whether the theorem of Pythagoras should be regarded as primarily an area theorem.

He disagreed with Mr. Durell's view that a mark on paper could be an actual line suitable for use in Stage B geometry; and preferred the ancient view that geometrical arguments refer, not to the marks made, but to the ideal lines which they represent or symbolise. Clear-cut assumptions are needed in deductive geometry: it is not enough to say that if AB is nearly equal to AC, then the angles ACB, ABC are nearly equal. Pupils rarely

raise the question of the nature of points and lines at an early stage: when they do, it may be not out of place to point out that, at the beginning of geometry or any other subject, it is necessary to take something for granted. The initial question is not how points and lines are to be defined, but what is to be assumed about them.

The discussion considered mainly the relative precedence of congruence and similarity to a child, the need for abstractions and assumptions, and the place of the proof of theorems in a geometry course.

There was fairly general agreement (though not quite unanimous) that the ideas of equality and congruence come to a child slightly earlier than those of approximate equality or similarity, whether the latter word is taken in the strict geometrical sense or in its popular sense. It was also agreed that, apart from this, children find it more difficult to understand geometrical similarity and proportion than congruence. Even numerical examples on similarity are hard for children, and general arguments very hard. It seems necessary to take congruence first if any deductive work is to be done, and some very easy examples on congruence are necessary, of the sort disparaged by Mr. Durell. It is desirable to prove the mid-point and intercept theorems by congruence. On the other hand, before coming to geometry children have been using maps for years in geography, so to prove formal similarity theorems is pedantic. A simple approach to similarity is through magnification of a plane figure by perspective from a point.

It was agreed that the basis of geometry is the physical world, of which the child has a large experience, though much of it unconscious. The study of geometry is to develop this experience and make it conscious. For this, some abstractions are necessary, and assumptions must be introduced as they are needed, but not a set list of assumptions. The child's " intuition " sometimes misleads him, and must be corrected by both graphical demonstration and deductive proof.

Several speakers agreed on the value of learning a few fundamental theorems. These show the pattern of a sustained argument containing several steps, and teach preciseness of expression. A child needs practice in writing out arguments, and theorems help him to pass from the stage of learning by memory and imitation to learning by reason. One speaker even said that children *like* learning theorems.

Finally, Prof. Christofferson stressed the importance of helping the child to transfer the pattern of geometrical proof to other branches of thought.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

SUR DES POINTS DE GERGONNE ET DE NAGEL D'UN TÉTRAÈDRE.

PAR VICTOR THÉBAULT.

Introduction. De nombreuses analogies entre les propriétés du triangle et du tétraèdre ont déjà été signalées qui concernent des points, des droites, des cercles et des sphères remarquables associés à ces deux configurations.*

Nous nous proposons de définir des points d'un tétraèdre qui possèdent des propriétés comparables à celles des points de Gergonne et de Nagel d'un triangle. Ceux-ci pouvant recevoir plus d'une définition, d'après les propriétés qu'ils possèdent, plusieurs points pourront leur correspondre dans le tétraèdre comme nous le montrerons pour le point de Nagel.

1. THÉORÈME. *Dans un triangle $T \equiv ABC$, les coordonnées barycentriques du point de Lemoine K sont inversement proportionnelles aux coordonnées normales de ce point par rapport au triangle tangentiel $T' \equiv A'B'C'$.*

En effet, a, b, c, R et m_a, m_b, m_c désignant les longueurs des côtés BC, CA, AB, du rayon du cercle circonscrit (O) et des médianes correspondant aux sommets A, B, C du triangle T, on obtient, d'abord,

$$AK = 2bcm_a/(a^2+b^2+c^2), \quad AA_1 = bc/m_a,$$

et des formules analogues pour BK et BB_1, CK et CC_1, A_1, B_1, C_1 étant les points où les symédianes AK, BK, CK recoupent le cercle (O).

D'autre part, si x, y, z représentent les distances des sommets A, B, C aux antiparallèles égales qui passent par le point K, on a

$$AK \,.\, AA_1 = 2b^2c^2/(a^2+b^2+c^2)$$
$$= 2a^2b^2c^2/a^2(a^2+b^2+c^2) = x \,.\, 2R\,;$$

d'où, et par analogie,

$$x \,.\, a^2 = y \,.\, b^2 = z \,.\, c^2, \qquad \text{(0)}$$

ce qui achève de démontrer le théorème.

COROLLAIRE. *Dans un triangle T, les coordonnées normales du point de Lemoine K_1 du triangle $T_1 \equiv A_1B_1C_1$ ayant pour sommets les points de contact du cercle inscrit (I), de centre I, avec les côtés BC, CA, AB sont proportionnelles aux quantités.*

$$\cos^2 \tfrac{1}{2}B \cos^2 \tfrac{1}{2}C, \quad \cos^2 \tfrac{1}{2}C \cos^2 \tfrac{1}{2}A, \quad \cos^2 \tfrac{1}{2}A \cos^2 \tfrac{1}{2}B. \qquad \text{(1)}$$

Car, d'après le théorème précédent, les coordonnées normales du point K_1 par rapport à T, qui sont inversement proportionnelles à $B_1C_1^2$, $C_1A_1^2$, $A_1B_1^2$ sont proportionnelles aux quantités (1). Ces quantités étant inversement proportionelles à $a(p-a)$, $b(p-b)$, $c(p-c)$, le point K_1 coïncide avec le point de Gergonne Γ du triangle T, ce qui est d'ailleurs évident puisque T est le triangle tangentiel de T_1.

On obtient, en outre,

$$ax \cot \tfrac{1}{2}A = by \cot \tfrac{1}{2}B = cz \cot \tfrac{1}{2}C. \qquad \text{(2)}$$

2. Le réciproque du point K_1, par rapport au triangle T, qui a pour coordonnées barycentriques $p-a$, $p-b$, $p-c$, se confond avec le point de Nagel, tandis que son conjugué isogonal dont les coordonnées normales sont proportionnelles à $a(p-a)$, $b(p-b)$, $c(p-c)$ coïncide avec le centre de similitude interne des cercles inscrit et circonscrit au triangle T.

* Cfr. les travaux de J. Neuberg, R. Bricard, N. A. Court, P. Delens, V. Thébault, etc.

3. Théorème. *Dans un tétraèdre $T \equiv ABCD$, les coordonnées barycentriques du second point de Lemoine L sont inversement proportionnelles aux coordonnées normales de ce point par rapport au tétraèdre tangentiel*

$$T' \equiv A'B'C'D'.$$

Nous désignerons, comme d'usage, les aires des faces BCD, CDA, DAB, ABC et le volume du tétraèdre T par A, B, C, D et V ; les longueurs des arêtes BC, DA, CA, DB, AB, DC et les dièdres suivant ces arêtes par a, a', b, b', c, c' ; les rayons des sphères inscrite et circonscrite (I) et (O) par r et R, et pour la commodité, nous poserons

$$aa' + bb' + cc' = s, \quad ab'c' + bc'a' + ca'b' + abc = S.$$

Si la symédiane DL recoupe la sphère (O) en D_1, on sait que *

$$DL \,.\, DD_1 = \frac{s}{S} \,.\, a'b'c' = t \,.\, 2R.$$

Il en résulte, et par analogie. les relations

$$x \,.\, ab'c' = y \,.\, bc'a' = z \,.\, ca'b' = t \,.\, abc,$$

x, y, z, t étant les distances des points A, B, C, D aux sections antiparallèles égales menées par le point L dans les trièdres (A), (B), (C), (D), et ces distances sont égales à celles du point L aux plans des faces correspondantes du tétraèdre T'. Le théorème résulte de ce que les coordonnées barycentriques du point L sont proportionnelles à $ab'c'$, $bc'a'$, $ca'b'$, abc. †

Corollaire. *Dans un tétraèdre T, les coordonnées normales du second point de Lemoine L_1 du tétraèdre $T_1 \equiv A_1B_1C_1D_1$ ayant pour sommets les points de contact de la sphère inscrite avec les faces BCD, CDA, DAB, ABC sont proportionnelles aux quantités.*

$$\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c', \quad \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c' \cos \tfrac{1}{2}a', \quad \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b',$$
$$\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c. \qquad (3)$$

En effet, d'après le théorème précédent, les coordonnées normales du point L_1, par rapport au tétraèdre T, sont proportionnelles aux produits

$$A_1B_1 \,.\, A_1C_1 \,.\, A_1D_1, \quad B_1C_1 \,.\, B_1D_1 \,.\, B_1A_1, \quad C_1D_1 \,.\, C_1A_1 \,.\, C_1B_1,$$
$$D_1A_1 \,.\, D_1B_1 \,.\, D_1C_1$$

des arêtes de T_1 et ces produits sont eux-mêmes proportionnels aux quantités (3).

4. Relations entre des Éléments d'un Tétraèdre. Modifiant légérement les notations, nous désignerons par x, y, z, t les coordonnées normales du point L_1 par rapport au tétraèdre $T \equiv ABCD$ pris comme tétraèdre fondamental. Les cotangentes des demi-angles α, β, γ, δ aux sommets des cônes de sommets A, B, C, D circonscrits à la sphère inscrite (I) à T, ont pour expressions

$$\cot \alpha = 4 \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a' / \Delta_a',$$
$$\cot \beta = 4 \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b' / \Delta_b',$$
$$\cot \gamma = 4 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c' / \Delta_c',$$
$$\cot \delta = 4 \cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c' / \Delta_d',$$

Δ_i', $(i = a, b, c, d)$, étant les sinus des trièdres supplémentaires des trièdres (A), (B), (C), (D).

* V. Thébault, *Bull. de la Soc. Math. de France*, 1948, 100.

† V. Thébault, *Ann. de la Soc. Scient. de Bruxelles*, 1922, 174.

Il en résulte que

$$AB_1 = AC_1 = AD_1 = r \cot \alpha = (8BCD/9V^2) r \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a',$$
$$BA_1 = BC_1 = BD_1 = r \cot \beta = (8CDA/9V^2) r \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b',$$
$$CA_1 = CB_1 = CD_1 = r \cot \gamma = (8DAB/9V^2) r \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c',$$
$$DA_1 = DB_1 = DC_1 = r \cot \delta = (8ABC/9V^2) r \cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c'. \quad \text{.......(4)}$$

D'autre part, comme Δ_a', Δ_b', Δ_c', Δ_d' sont proportionnels à A, B, C, D, on obtient

$$A \cot \alpha : B \cot \beta : C \cot \gamma : D \cot \delta =$$
$$\cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a' : \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b' :$$
$$\cos \tfrac{1}{2}c \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b' : \cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c'. \quad \text{....................(5)}$$

5. En tenant compte des expressions (3) des coordonnées x, y, z, t, on obtient les relations

$$Ax \cot \alpha = (8ABCD/9V^2) \cos \tfrac{1}{2}a \cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c \cos \tfrac{1}{2}c'$$
$$= By \cot \beta = Cz \cot \gamma = Dt \cot \delta, \quad \text{......................................(6)}$$

comparables aux égalités (0). D'où cette proposition *

THÉORÈME. *Les coordonnées barycentriques du second point de Lemoine* L_1 *du tétraèdre* T_1, *par rapport au tétraèdre fondamental* T, *sont proportionnelles aux tangentes des angles* α, β, γ, δ *que font les bissectrices* AI, BI, CI, DI *des trièdres* (A), (B), (C), (D) *avec les faces adjacentes.*

Cette propriété constitue une analogie avec le triangle et nous proposons de dire que le point $L_1 \equiv \Gamma$ se confond avec un point de Gergonne du tétraèdre T.

6. Par rapport au tétraèdre fondamental T, le réciproque N du point $L_1 \equiv \Gamma$ dont les coordonnées barycentriques sont proportionnelles à

$$1/A \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c' \quad \text{ou à} \quad \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \tfrac{1}{2}a'/A, \ldots$$

est le point de Nagel correspondant au point Γ.

7. AUTRE POINT DE NAGEL D'UN TÉTRAÈDRE. Si les cercles exinscrits (I_a), (I_b), (I_c) d'un triangle ABC touchent les côtés BC, CA, AB en D_a, E_b, F_c, on a visiblement

$$BC + CE_b = CA + AF_c = AB + BD_a = \tfrac{1}{2}(BC + CA + AB).$$

Les céviennes AD_a, BE_b, CF_c concourent au point de Nagel du triangle ABC qui se confond avec le centre du cercle inscrit au triangle anticomplémentaire de celui-ci.

L'analogie suivante a lieu dans un tétraèdre $T \equiv ABCD$.†

THÉORÈME. *Dans un tétraèdre* T, *les droites* AA', BB', CC', DD' *qui joignent les sommets aux points* A', B', C', D' *des faces opposées tels que l'on ait les relations d'aires*

$$CDA + CDA' = DBA + DBA' = BCA + BCA' = \tfrac{1}{3}(A + B + C + D), \ldots,$$

concourent au centre N' *de la sphère inscrite au tétraèdre anticomplémentaire.*

Le point N' est un autre point de Nagel du tétraèdre fondamental.

8. En remplaçant la sphère inscrite (I) par l'une des *sept* sphères tangentes aux quatre plans des faces du tétraèdre T on obtient des relations analogues à (4), (5), (6) qui déterminent sept points *associés* au point $L_1 \equiv \Gamma$ comparables aux trois associés du point de Gergonne d'un triangle et la conclusion est la même pour ce qui concerne les points de Nagel N et N'. V. T.

* R. Bouvaist, *Mathesis*, t. 54 (Supplément, p. 19).

† V. Thébault, *Mathesis*, t. 56, pp. 64 et 257, question 3323.

SOME CHARACTERISTIC PROPERTIES OF THE CIRCLE.

By R. A. Rosenbaum.

There is a considerable literature on characteristic properties of the circle; a bibliography is listed in Bonnesen-Fenchel [1]. It is proposed here to present some simple characteristic properties of types which are, in a sense, related to one another, as will be seen in what follows.

Principal Definitions and Theorems. The discussion will be restricted to smooth, convex, closed curves. Relative to such a curve, a point, P, will be called

(i) a π-point, if the product of the segments of all chords through P is constant;

(ii) an α-point, if, for every chord through P, the angles, on the same side of the chord, between the chord and the tangents to the curve at the extremities of the chord are equal; and

(iii) a β-point, if the perpendicular bisectors of all chords through P are concurrent.

Theorem 1. A π-point within a curve is an α-point, and conversely.

Proof. Suppose that P is a π-point, and that MPN, XPY are two chords through P. Then, from similar triangles, $\angle MXP = \angle YNP$. This equality is preserved as X approaches M along the curve; so that, if TMT', UNU' are tangents to the curves at M, N, with T and U on the same side of MN, it follows that $\angle TMP = \angle UNP$.

To prove the converse, we start with a segment MPN and a smooth convex arc MXN. In attempting to obtain a smooth convex arc MYN which, together with the first arc, forms a smooth convex curve with respect to which P is an α-point, we are led to the solution of an equation $\dfrac{dy}{dx} = f(x, y)$, where $f(x, y)$ satisfies a Lipschitz condition. Hence, there is a unique solution. But the arc MYN obtained from the arc MXN by imposing the condition that P be a π-point is a solution. This completes the proof.

It is clear now that, in a theorem relating to smooth convex curves, the conditions of an interior point's being a π-point or an α-point are interchangeable. In particular, there is a theorem (Yanagihara [3]) which states that, if there are two π-points within a curve, the curve is a circle. Hence, two interior α-points are also characteristic of a circle.

There is no significance to a π-point *on* a curve, but there are results for α-points on a curve. Rademacher and Toeplitz [2] prove very neatly that, if every point of a smooth closed curve is an α-point, then the curve is a circle. Their proof would work equally well under the assumption merely that there exist two α-points on the curve. We can go a step farther and prove:

Theorem 2. If there exists an α-point on a smooth convex closed curve, then the curve is a circle.

Proof. Suppose that the α-point, P, is chosen as the pole, and the tangent at P as the polar axis, of a polar coordinate system. Then the assumption that P is an α-point implies that $\tan\theta = r \Big/ \dfrac{dr}{d\theta}$, *i.e.* that $r = c\sin\theta$.

As the next three theorems show, two "independent" conditions with respect to π- and β-points determine a circle.

Theorem 3. If there exist two β-points within a closed convex curve, then the curve is a circle.

Proof. Suppose that the perpendicular bisectors of chords through the

β-points P, Q concur in O_p, O_q. Let the extremities of the chord PQ be M and N; to fix the ideas, suppose that MN is horizontal and that the points, reading from left to right, are M, P, Q, N, and that Op is above Oq. Choose any point R on the curve on the same side of MN as Op, and draw a circle with Op as centre, OpR as radius. Draw the chord RP and extend it to X on the curve. Since Op is the centre of the circle and also the point of concurrency of perpendicular bisectors of chords through P, X_1 lies on the circle as well as on the given curve. Draw X_1Q and extend to X_2 on the curve. Since Oq is below Op, the foot of the perpendicular from Oq to X_1Q is nearer to X_1 than is the foot of the perpendicular from Op to X_1Q. Therefore, X_2 lies inside the circle. Draw X_2P and extend to X_3 on the curve; this point is also within the circle. Continuing in this fashion, we obtain a sequence of points, X_2, X_4, ..., all inside the circle and approaching the line MN from above. Similarly, the sequence X_3, X_5, ... lies within the circle and approaches the line MN from below. Now draw RQ and extend to Y_1 on the curve, draw Y_1P and extend to Y_2 on the curve, etc. By the same sort of argument that has already been used, we see that Y_1, Y_3 ... are outside the circle and approach MN from below, while Y_2, Y_4, ... are outside the circle and approach MN from above. A consideration of all the sequences shows that the circle must pass through M and N. Starting with another point S of the curve would lead to the same circle: centre Op, passing through M and N. Hence, all points of the curve lie on the same circle, and Op coincides with Oq. (A modification of some of the steps would be required if, as might happen, X_2, for example, were below MN. But the method of proof would still apply.)

Theorem 4. If, within a smooth convex curve, there exists a point, P, which is both a π-point and a β-point, then the curve is a circle.

Proof. Suppose that the perpendicular bisectors of all chords through P meet at O. Let X be any point on the curve, and draw the circle with centre O, radius OX. Draw the chord XP and extend it to Y on the curve. Then Y is also on the circle. Suppose that some chord, UPV, of the curve meets the circle in U' and V'. Then $UP \cdot PV = XP \cdot PY = U'P \cdot PV'$. But also, if U' is inside (outside) the curve, so is V'. These conditions can be satisfied only if $U = U'$, $V = V'$; *i.e.* if all points of the curve lie on the circle.

Theorem 5. If there exists a π-point, P, and a β-point, B, within a smooth, convex, closed curve, $\mathfrak{C}$, then $\mathfrak{C}$ is a circle.

Outline of a (cumbersome) proof. Suppose that the perpendicular bisectors of chords through B meet at O_b, and that the chord PB meets $\mathfrak{C}$ in M and N. Draw the circle, centred at O_b, passing through M and N. If $\mathfrak{C}$ does not coincide with this circle, $\mathfrak{C}$ must intersect it—indeed, infinitely often. For, if X is a point of $\mathfrak{C}$ outside the circle, and if XPX_1, X_1BX_2 are chords of $\mathfrak{C}$, then X_1, X_2 lie within the circle, etc. Let U, V be two points of intersection of $\mathfrak{C}$ and the circle, with all points of $\mathfrak{C}$ between U and V lying outside the circle. Then the arc of $\mathfrak{C}$ between U_1 and V_1, where UPU_1, VPV_1 are chords of $\mathfrak{C}$, lies within the circle. Next, the arc of $\mathfrak{C}$ between U_2 and V_2, where U_1BU_2, V_1BV_2 are chords of $\mathfrak{C}$, lies within the circle, etc.

Now, starting with the arc UV of $\mathfrak{C}$, we can construct a curve, $\mathfrak{C}'$, through M and N for which P and B are both π-points. It can be shown that, for all $n > n_0$, the arc of $\mathfrak{C}$ between Un and Vn has points which are farther from the circle than are any points of $\mathfrak{C}'$ between Un and Vn. But, from Yanagihara's theorem, we know that $\mathfrak{C}'$ has no tangents at M and N; for, if it did, $\mathfrak{C}'$ would coincide with the circle. Hence, *a fortiori*, $\mathfrak{C}$ is not a smooth curve, against the hypothesis.

Extensions and Related Results. For the preceding work, with the exception of the second half of Theorem 1, the assumption of convexity can be weakened to an assumption of star-likeness with respect to the distinguished

point or points involved. On the other hand, convexity implies the existence of at least one-sided tangents at each point of the curve ; and the wording of the definitions and proofs can be changed so as to make one-sided tangents adequate ; hence, if convexity is assumed, the assumption of smoothness is superfluous.

One might ask whether the condition that the distinguished point or points lie within the curve could not be removed. Yanagihara's proof for the case of two π-points, for example, may be modified so as to give the same result when one of the points is inside, and the other outside, the given curve. In case both points are outside the curve, and their chord intersects the curve in two distinct points, then the same method of proof shows that the curve is a circle, or consists of the arcs of two circles. (Clearly, the assumption of smoothness has been dropped here.) If the chord is tangent to the curve or does not meet the curve, Yanagihara's mode of attack fails. The solution of this problem does not seem to be known.

Other types of distinguished points may be considered. For example, relative to a closed convex curve, a point P may be called :

(iii′) a β'-point, if all chords through P are bisected at P ; and

(iv) a λ-point, if all chords through P have the same length.

Then it is trivial that, if there exists a point which is at once a β'-point and a λ-point, the curve is a circle. It is simple to show that, if there exists a point which is both a π-point and a λ-point, the curve is a circle. The cases of a point which is both a β'-point and a π-point, and of two points, one a β'-point and the other a π-point, are special cases of Theorems 4 and 5. Whether a convex curve with 2 λ-points can exist is a well-known unsolved problem. The remaining combinations of points of types considered above also appear to be difficult to handle.

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2. Rademacher, Toeplitz, *Von Zahlen und Figuren*.
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GLEANINGS FAR AND NEAR.

1626. About Carmenta we know from the historian Dionysus Periergetes that she gave oracles to Hercules and lived to the age of 110 years. 110 was a canonical number, the ideal age which every Egyptian wished to reach, and the age at which, for example, the patriarch Joseph died. The 110 years were made up of twenty-two Etruscan lustra of five years each ; and 110 years composed the " cycle " taken over from the Etruscans by the Romans. At the end of each cycle they corrected irregularities in the solar calendar by intercalation, and held Saecular Games. The secret sense of 22—sacred numbers were never chosen haphazardly—is that it is the measure of the circumference of the circle when the diameter is 7. This proportion, now known as pi, is no longer a religious secret ; and is used today only as a rule of thumb formula, the real mathematical value of pi being a decimal figure which nobody has yet been able to work out because it goes on without ever ending, as 22/7 does, in a neat recurrent sequence.—Robert Graves, *The White Goddess*, Second Edition, Chapter XIII, p. 208. [Per Mr. J. E. Blamey.]

MATHEMATICAL NOTES.

2068. *Postage stamp portraits.*

With reference to Note 1682, some further examples are :

France,	1937 ;	Descartes, 90 c.
Vichy	1944 ;	Pascal, 1.20 + 2.80.
Eire,	1943 ;	Hamilton, ½d., 2½d.
Denmark,	1944 ;	Rømer, 20 øre.
Denmark,	1946 ;	Tycho Brahe, 20 øre.
Italy,	1942 ;	Galileo, 10 c., 25 c., 50 c., 1 l. 25 c. (four different designs).
Russia,	1946 ;	Chebichev, 30 k., 60 k.

The General Government of Poland provided a fine portrait of Copernicus on a green stamp. C. B. GORDON.

[Interested readers may care to have a reference to the recent detailed article on this topic by Carl B. Boyer, in *Scripta Mathematica*, Vol. XV, No. 2, pp. 105–114, illustrated by a number of excellent plates.]

2069. *An extension of Simpson's Rule.*

Given the values of $f(x)$ for $x=a,\ a+h,\ a+2h, \ldots a+nh$, it is possible to obtain an approximate value of

$$\int_a^{a+nh} f(x)\,dx$$

by means of Simpson's rule, provided n is even.

As problems in which n is odd are just as likely to occur as those in which n is even, it is important to be able to extend the rule to cover this other case ; perhaps the following is the simplest procedure.

In deriving Simpson's rule we show that the parabola

$$y=Ax^2+Bx+C$$

passes through the points $(-h, y_1)$, $(0, y_2)$, (h, y_3) if

$$2Ah^2=y_1-2y_2+y_3, \quad 2Bh=y_3-y_1, \quad C=y_2,$$

and then show that

$$\int_{-h}^{h}(Ax^2+Bx+C)\,dx=\tfrac{1}{3}h(y_1+4y_2+y_3).$$

If, instead, we evaluate over the range $(-h, 0)$ we obtain

$$\int_{-h}^{0}(Ax^2+Bx+C)\,dx=\tfrac{1}{3}h(\tfrac{5}{4}y_1+2y_2-\tfrac{1}{4}y_3),$$

giving the following extension of Simpson's rule :

To obtain an approximate value of

$$\int_a^{a+nh} f(x)\,dx,$$

when n is odd, evaluate the integral between $x=a+h$ and $x=a+nh$ by Simpson's rule and add

$$\tfrac{1}{3}h\{\tfrac{5}{4}f(a)+2f(a+h)-\tfrac{1}{4}f(a+2h)\}.$$

J. G. FREEMAN.

2070. *Wanted, a connection.*

Let S be a quadric and P_1 a given point in general position. Consider the equation (standard notation)

$$U_1^2S - 2U_1US_1 + U^2S_{11} = 0.$$

If $U=0$ is a plane, the equation determines the cone joining P_1 to the conic in which U, S intersect.

If $U=0$ is a quadric, the equation determines the quartic surface which is the locus of a point B defined as follows: the points A, B are conjugate with respect to S and collinear with P_1, and A lies on U.

[The first result is standard. The second result, based on a generalisation of inversion going back, I think, to Hirst in 1865, was new to me in this analytical form. Both interpretations can, of course, be extended to space of n dimensions.]

The object of this note is to ask whether there is any connection between these two interpretations of the equation. I derived the second and happened to recognise the form, but in mathematics one equation will not often do two jobs without just cause. E. A. MAXWELL.

2071. *On " cross " and " signed angle ".*

Professor Forder (*Gazette*, October 1947) has made a contribution of quite exceptional interest to this important geometrical question. (Peculiarly interesting to me is the conjunction of Area with Angle, in relations he finds fundamental for Abstract Geometry: for it was discussion of the former *—in university lectures, more than forty years ago—that led me on to examine, with care, in the same context of " sign ", the more obscure question of Angle.)

The purpose of this note is to discuss a point which has bothered me, since I first saw Forder's treatment of the question in his *Higher Course Geometry*: that it is expressed so much more *quantitatively* than my own. The *Gazette* article makes his position in this matter much clearer to me.

My initial approach was quantitative—in terms of positive and negative angles (see XI, 161, p. 188, § 2). But behind *quantity* lies *equality*; and fundamental to *equality* is *congruence*—primarily of Straight Line and Angle *figures*.† Hence, expression in terms of congruence—rather than in terms of quantity, when either is appropriate—seems to be preferable. (We are here concerned with the somewhat subtle basic case in which the " equal in every respect ", of congruence, means, in fact, equal in *only one* respect.)‡

I, therefore, adhere quite strongly to the scheme of notation set out in XI, p. 188—to the use of " Complete Angle ", rather than " cross ", as appropriate alternative to " line-pair " §—and to the view that the line-pair (or Complete Angle) *congruence* is the characteristic Angle-proposition of

* Of which the central proposition is that: *if A, B, C, ..., K, L be (any) given coplanar points, and P variable in the plane, the sum of (" signed ") areas*

$$\triangle PAB + \triangle PBC + \ldots + \triangle PKL + \triangle PLA$$

is constant; whence generalized definition of *area of closed polygon ABC ... KLA*—and, so, of *area of* (any) *closed plane curve.*

† Based upon the axiomatic congruence characteristics (1) of Straight Line, (2) of Plane—both of which are involved in congruence of Angles.

‡ And " equal " implies " congruent " (as it does not necessarily, in general).

§ See XI, 166, p. 385 (Note 684). The distinction (of initial letters) between *Angle*—for the *figure*—and *angle*—for the *quantity*—is, in particular, a very useful one. The Complete Angle (or line-pair) as a *figure*, associated with a *multiplicity* of angle-quantities, is the subject of this Note.

Euclidean Plane Geometry (more particularly, $PQR \equiv UVW$, for *every* pair of 3-permutations of four given co-planar points A, B, C, D).*

But I now see that I have underestimated the importance of the *quantitative* forms in the theory ; and that the alternative presentation in terms of them—although (as I see it) essentially less simple and elementary (because of the many-valuedness of " angle of inclination " †)—is, for certain purposes, appropriate (as, in particular, for the main purpose of the October 1947 article).

The elementary *angle* fact of the line-pair is that a line-pair is *specifiable quantitatively* by a " *signed angle* " *between* $\pm R$; ‡ and, if it were feasible to *confine* the specification to such (positive or negative acute) angles, the two modes of presentation would be equally simple—and the quantitative, therefore, the more useful (*because* quantitative). But it is just because the simple relation of congruence implies a *common multiplicity of angles of inclination*, that it gives what seems to be the essential mode of presentation—to which the alternative (however important) is secondary.

I wish, however, to close on a note of appreciation rather than of criticism. It is because Professor Forder is an authority on Geometry (and exhibited as such in the present context) that one reads very critically what he writes.

Postscript.

Since the above Note was posted to the Editor, the issues have cleared themselves (in an attempt to write them up for Professor Forder).

1. It is now clear to me that the Angle-theory (as required, more particularly, for pp. 227–9 of Forder's paper) takes its simplest and most elementary form in terms of the line-pair *congruence*. All that seems necessary is to use a non-quantitative (" Addition ") form in the basic proposition (xi, pp. 189-90)

$$(l_1, l_2) + (l_2, l_3) \equiv (l_1, l_3)$$

with immediate extension to n lines—without any need to require concurrency or the ignoring of " multiples of two right angles " : it is simply a properly defined use of the sign " + ", in this context—which is consistent with its quantitative use in the same context. Writing this in the alternative (congruence) form

$$(l_1, l_2) + (l_2, l_3) + (l_3, l_1) \equiv 0$$

—properly defining this use of the sign " $\equiv$ "—it simply expresses, in terms of congruences, completion of the cycle of lines. This gives at once (for lines CA, AB, BC) the triangle proposition

$$BAC + CBA + ACB \equiv 0$$

—as a *congruence* proposition (underlying the quantitative propositions), and there is immediate extension to n lines : in particular, for four lines,

$$ABC + BCD + CDA + DAB \equiv 0.$$

* See XI, 161, p. 191 (§ 9), and—especially—*Proc. London Math. Soc.*, Ser. 2, 23, p. 46.

† Note the important qualification (XXXI, 296, p. 227, line 8) : " multiples of two right angles are ignored."

‡ R denoting " right angle ". " *Signed* angle " seems to me preferable to " *directed* angle "—seeing (especially) that angles are *not* vector quantities (as defined in terms of *length*-vectors) ; but of more significance for this discussion is the passing comment that " crosses . . . do not distinguish between an angle and its supplement " (p. 231, § 5) : on the face of it that seems to be a return to the old ambiguity of " equal or supplementary ", from criticism of which my (above mentioned) initial quantitative approach was made, to this whole question (XI, p. 188, § 2)—but that is obviously not what Professor Forder means ; perhaps it is a question of the definition of " supplement "—possibly of the distinction between figure (Angle) and quantity (angle).

This, without any express reference to angle-*quantities*, appears to be the *elementary background* for § 2 of Forder's paper.

2. The (quantitative) angle-sum theorem (which underlies Ax. II′ in § 5) is perhaps worth looking at in this (rather obvious) way: the angles of inclination of two intersecting lines are such that two of them—say α, α'—are *positive and related by*

$$\alpha + \alpha' = S.$$

In terms of these can be specified the " interior " and " exterior " angles of a triangle (or other convex polygon); and *the latter* being determined from *one* direction of each line (in each of its two cases), their sum

$$A' + B' + \ldots = \text{rev. } \angle = 2 \,.\, S\,;$$

but

$$A + A' = S = B + B' = \ldots\,;$$

hence, for triangle ABC,

$$A + B + C = 3 \,.\, S - 2 \,.\, S = S\,;$$

and, for convex n-gon,

$$A + B + \ldots + L = n \,.\, S - 2 \,.\, S = (n-2) \,.\, S.$$

(The point is that the *exterior* angles have the simpler general property.)

3. I am very grateful to Professor Forder for stimulus to think more clearly on an important question, which is of great interest to me. His reference (p. 231) to my " difficulty " about order of points on the Circle (see Note 1899, May 1946—where the problem is reduced to what appear to be its simplest terms) stimulates me, more particularly, to write up commonsense axioms of Elementary Euclidean Geometry, which have been lying incomplete in old lecture notes for nearly forty years. That would make my contribution to this discussion the elementary counterpart of Forder's " higher " contribution. (I would like to stress, to the Association, the immense importance to Mathematics of " the elements "—when dealt with thoroughly on their true merits.)

D. K. Picken.

2072. *The probability integral.*

In a recent note * Mr. J. H. Cadwell has shown that the probability integral $\int_0^\infty e^{-x^2}dx$ can be evaluated by means of Cauchy's residue theorem. His method depends on performing two successive contour integrations, one round a rectangle and the other round the sector of a circle.

It may be worth mentioning, however, that the integral in question can, in fact, be obtained by a single contour integration. For consider the integral of $F(z) = e^{i\pi z^2}/\sin \pi z$ taken round the parallelogram having vertices at $R \pm \frac{1}{2} + iR$, $-R \pm \frac{1}{2} - iR$. The integrand has a simple pole of residue $1/\pi$ at $z=0$ and is otherwise regular on and within the contour of integration. It is easily verified that the integrals along the horizontal sides of the parallelogram tend to zero as $R \to \infty$. Furthermore, the sum of the integrals along the sides inclined to the real axis is

$$\int_{-R\sqrt{2}}^{R\sqrt{2}} \{F(te^{\frac{1}{4}i\pi} + \tfrac{1}{2}) - F(te^{\frac{1}{4}i\pi} - \tfrac{1}{2})\} e^{\frac{1}{4}i\pi} dt = 2i \int_{-R\sqrt{2}}^{R\sqrt{2}} e^{-\pi t^2} dt.$$

Making use of Cauchy's residue theorem, and proceeding to the limit as $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1, \quad \textit{i.e.} \int_0^\infty e^{-x^2} dx = \tfrac{1}{2}\sqrt{\pi}.$$

L. Mirsky.

* *Math. Gazette*, XXXI, Note 1987, (October, 1947).

2073. *The school mathematics laboratory.*

Great progress has been achieved in the teaching of school mathematics during the last thirty years. We have seen introduced a greater degree of integration on the subject itself, a less academic approach in its presentation to the scholars, and a more frequent and more varied use of visual aids. But the position even now is far from satisfactory. For the bulk of the pupils leave the grammar schools with too slight a knowledge of the subject and even less enthusiasm for it. If that is the result of the labours of the best mathematical teachers upon the intellectual cream of the child population, how dismal must be the outlook for the lesser lights of our modern and technical schools!

Recognition of this situation too often results in a defeatist attitude among teachers of mathematics in the modern and technical schools leading to the abandonment of any attempt to teach anything very substantial beyond Arithmetic, Graphs, and a little Practical Geometry. Another reaction to this situation is to recognise in it the implicit challenge to our mathematical teaching methods. Let us take up this challenge and see if we can suggest a change in methods which would not only save the modern and technical children from the blight of mathematical ignorance, but also shed light in the darker corners of our grammar schools.

Before discussing what changes are necessary, let us state the aim of secondary school mathematics. In my opinion, the purpose of mathematical teaching in the secondary school should be to increase and *widen* the range of the scholar's mathematical knowledge, so that he can deal more intelligently with the quantitative and spatial features of the world around him. This implies not only the learning of a considerable body of exact knowledge covering a wide field of mathematics, but also a realisation on the scholar's part that although mathematics can be made to look dismayingly abstract, it nevertheless has its roots in the real world and has arisen from the needs of man to measure land, capacity, time, motion, etc. As Mr. F. W. Westaway puts it: "Below the Sixth Form, mathematics is essentially a *practical* subject (Westaway's emphasis), not a subject for philosophic speculation. Never press forward formal abstract considerations until experience has paved the way." *

Most experienced teachers will agree that mathematics is best taught when approached practically, but surely, there are very severe restrictions placed upon this method if the subject is always taught in a room which lacks the facilities and equipment of a mathematical laboratory. Fearing, I suppose, that the stigma of an experimental subject might attach to mathematics with a certain loss of caste on that account, the idea of setting apart a room as a mathematical laboratory has only been timidly suggested and, so far as I know (and I have made many inquiries among members of the Mathematical Association), never actually been realised. The *Handbook of Suggestions to Teachers* seems to envisage this kind of thing without actually specifying it. Even Mr. W. L. Sumner in his excellent book, *The Teaching of Arithmetic and Elementary Mathematics*, devotes only a few pages to describing in a tentative way the equipment and function of a mathematical laboratory.

There are, however, plenty of mathematical laboratories attached to universities and research stations. Professor H. Levy, referring to his laboratory at the Imperial College, wrote in the *Gazette*, Vol. XII, p. 374:

"There is no doubt whatever that graphical and arithmetical methods, especially the former, are potent means of quickening and stimulating the mathematical interests of students, who would otherwise be classified as

* Westaway, *Craftsmanship in the Teaching of Elementary Mathematics*, p. 11.

stupid ; and many of these students by this process have been led back to the more pure, and possibly the more fundamental, branches of mathematics by the insight they have acquired from their training in the mathematical laboratory."

If a mathematical laboratory has proved so helpful to students at university level, then surely it could prove at least as beneficial to youngsters finding their feet in the subject at school. It is therefore my contention that a mathematical laboratory should be an essential feature of any secondary school in which a systematic course of experimental and practical work would be carried out in conjunction with the mathematical syllabus.

Let us describe briefly some of the desirable features and equipment of a school mathematics laboratory together with some of its uses.

The room itself should be large and spacious, capable of being blacked out, supplied with the fittings usually found in an elementary mechanics laboratory, and having plenty of cupboard room and showcases for the display of models.

It should contain the usual measuring apparatus of an elementary physics laboratory, such as verniers, slide rules, micrometer screwgauges, planimeter, spherometers, measuring cylinders, sextants, clinometers, U-tubes, stop-clocks, direct reading Boyle's Law apparatus, Hooke's Law apparatus, etc.

Some of its uses would be :

1. The measurement of lengths, areas, volumes, weights and densities of actual objects ; also heights and distances.

2. The construction of cardboard models of three-dimensional figures from nets ; "Juneero" parts supplemented by coloured plastic strings have been found particularly useful for making models whose interior needs to be seen.

3. The introduction to the use of tables, slide rules, comptometer, and multiplying machines.

4. The experimental verification of such laws as that of Levers, Parallelogram of Vectors, Hooke's Law, and experimental geometry, including work on loci and envelopes and symmetry.

5. Scale drawing and graphs should be carried out in conjunction with measurements actually performed under 4. Statistical graphs could also be made from height measurements taken of a certain age group of the school by the pupils themselves. In such a laboratory it would be easy to bring into action the full battery of visual aids such as the films, models, charts and the use of shadows ; which is not the case if the mathematics teacher is expected to work in any odd room.

The course of practical work would, of course, be coordinated with the mathematics syllabus followed in the school. In the modern and technical schools which are free from the fetters of an external examination, it should be possible with the aid of such a powerful instrument as a mathematics laboratory to conduct interesting experiments in the recasting of mathematical syllabi which might have important results for the whole field of secondary mathematical education. In a future article I hope to give a more detailed account of the results of the work done in the laboratory now being set up in the school at which I work. B. Evans.

2074. *Definition of indices.*

Is there any reason against defining x^5 (say) as 1 multiplied by x five times :

$$x^5 = 1 \times x \times x \times x \times x \times x?$$

This method of definition will enable the pupil easily to avoid incorrect statements such as "$x^2 = x$ multiplied by itself twice" ; and the interpretation of x^0 as 1 seems to follow more naturally from this definition. B. Evans.

2075. *On nearly-isosceles right-angled triangles.*

The problem of finding right-angled triangles with sides of integer length is an old one. The well-known identity

$$(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$$

gives as many as desired, by inserting values for x and y.

Recently I noticed a way of finding such triangles which have their two shortest sides consecutive integers (see any work on Continued Fractions). They are made from the convergents to $\sqrt{2}$:

$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \tfrac{41}{29},\ \tfrac{99}{70},\ \tfrac{239}{169},\ \text{etc.}$$

Take any odd convergent. The denominator gives the length of the hypotenuse of the triangle ; the numerator split into consecutive integers gives the other two sides. The odd convergents above give the Pythagoras cases : 1, 1, 0 ; 3, 4, 5 ; 29, 21, 20 ; 169, 120, 119. Thus a series of right-angled triangles is generated, which in proportion get more and more nearly isosceles.

The same series of triangles is given by the above identity by substituting for y and x any two consecutive denominators of the convergents.

It has been pointed out to me that the numerators of the convergents are all atomic weights of radioactive isotopes. Whatever next? B. D. P.

2076. *A note on change-ringing.*

With reference to the Chain Rule (see article in *Gazette* No. 297), I have known the rule since June 1946 under another guise.

When bell-ringers ring long series (" touches ") of permutations (" changes "), to ease the task of remembering what is to happen next touches are composed in which a cycle of operations is repeated a number of times. Thus there occur certain permutations at regular intervals which finish the various similarly arranged parts of the touch. For instance, on 8 bells these part-ends might be :

1	2	3	4	5	6	7	8
5	1	8	3	2	7	4	6
2	5	6	8	1	4	3	7
1	2	7	6	5	3	8	4

etc., finishing in total of 15 operations.

The same series of operations produces the same " transpositions " between these consecutive part-ends. It is easily seen that the bells 1, 2, 5 will repeat each others' work cyclically and correspond to a 3-linked chain of Messrs. Chater. Also 3, 8, 6, 7, 4 form a 5-linked chain. I have termed these sets of mutually repeating bells " cycles ", and my rule is :

" The parity of a change may be determined by adding the number of bells to the number of cycles."

This rule is identical with the chain rule. As there were many excellent mathematicians interested in composition of peals in the last century this rule is probably old. I have proved it.

It is interesting to note that the oddness or evenness of permutations assumes great importance in ringing theory, and has long been known to bell-ringers as " In and Out of Course ". B. D. Price.

2077. *The locus of the Frégier point for a parabola touches the evolute.*

It is known that, if a variable chord of a conic subtends a right angle at a point P of the curve, all such chords pass through a fixed point F (the Frégier

point of P) on the normal at P. If the conic is the parabola $y^2 = 4ax$ and P is $(at^2, 2at)$, then the coordinates of F are $a(t^2+4)$, $-2at$. Thus the locus of F is the parabola $y^2 = 4a(x-4a)$.

The evolute of the original parabola is $4(x-2a)^3 = 27ay^2$. By solving the last two equations, we find that the Frégier parabola and the evolute touch at the points $(5a, \pm 2a)$. These are the Frégier points (and centres of curvature) of the ends of the latus rectum of the original parabola. J. BUCHANAN.

2078. *On Note* 1996: *definition of logarithm.*

Although I do not entirely agree with Mr. Lyness about $\log x$, I think that it should be said that Note 1996 only deals with one of the objections to the integral method, and perhaps the least important objection.

The two formulae

$$\text{(i)}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \quad \text{(ii)}\ \int z\,du = \int z\frac{du}{dx}\,dx,$$

are at the same level. On the other hand, many students become familiar with (i) before they reach (ii). For such students the best procedure is as follows:

$$\frac{d}{dx}\{\text{hyp}\,(cx) - \text{hyp}\,x\} = \frac{c}{cx} - \frac{1}{x} = 0\,;$$

hence

$$\text{hyp}\,(cx) - \text{hyp}\,x = \text{constant} = \text{hyp}\,c,$$

so that the use of (ii) is avoided. A. R.

2079. *The functional relations of the logarithmic and exponential functions.*

The account given of these functions in the article by Mr. Gant, Vol. XXX, No. 292, p. 277, is an interesting one. The introduction of the formal treatment of these functions by means of the definite integral definition of $\log x$, or the equivalent statement in the article cited, is in my view the easiest and most interesting line of approach for the beginner. But the necessary transformation of the definite integral by which the law $\log(ab) = \log a + \log b$ is obtained proves a stumbling-block in a number of cases. All the more interest therefore attaches to the alternative treatment given by Mr. Gant. But surely this is more difficult than it need be. This law, and the corresponding law for the exponential function, can be established more simply as follows:

(1) To prove that if $L'(x) = 1/x$, $L(1) = 0$, then $L(a) + L(b) = L(ab)$ for all positive a and b.

Consider $f(x) = L(a) + L(x) - L(ax)$.

Then $f'(x) = 1/x - a/ax = 0$, so that $f(x)$ is a constant.

On putting $x = 1$, the constant is seen to be $L(1) = 0$, and the result follows with $x = b$.

(2) To prove that if $E'(x) = E(x)$, $E(0) = 1$, then $E(a)\,.\,E(b) = E(a+b)$ for all a and b.

Consider $f(x) = E(x+a)/E(x)$, since $E(x)$ is never zero.

Then $f'(x) = \{E(x)\,.\,E(x+a) - E(x+a)\,.\,E(x)\}/\{E(x)\}^2 = 0$.

Therefore $f(x)$ is a constant, and on putting $x = 0$ it is seen to be $E(a)$. The result follows with $x = b$. H. MARTYN CUNDY.

2080. *Note on Robert Record.*

As so little is known about Record this extract may be interesting. It is taken from *The Ordinall of Alchemy* by Thomas Norton of Bristol (reproduced from *Theatrum Chemicum Britannicum* with annotations by Elias Ashmole)

with introduction by E. J. Holmyard, M.A., D.Litt., and is one of Ashmole's notes on Norton's *Alchemy* :

"(*c*) John Pitts from John Bale, and (*d*) he from Robert Record, relates, that this Thomas Norton, was *Alchymista suo tempore peritissimus*, and much more curious in the Studies of *Philosophy* than others, yet they passe some undecent and abusive *censures* upon him, with reference to this *vaine and frivolous Science*, as they are pleased to tearme it, (and a better opinion I find not they had even of the *Hermetic learning* it selfe)."

So Record, unlike his contemporary Dee, was not attracted to Alchemy. It would be interesting to know if these two ever met. Record received an M.D. at Cambridge in 1545, and Dee a B.A. in 1546. R. S. WILLIAMSON.

2081. *On Note* 1918.

The following are two further methods of constructing a circle to touch the arc and the two bounding radii of a quadrant, the figure in the above note being used and it being assumed that the circle of which ACB is an arc is completed.

(1) Produce AO to cut the circle in A', BO to cut it in B'. Join CA', CB' to cut OB, OA in K and L respectively. Then K and L are the points of contact of the required circle with these lines. Further, if AC, BC be produced to cut OB produced and OA produced in K and L respectively, these are the points of contact of the circle drawn to touch the arc and the bounding radii produced.

These constructions are of interest as being a particular case of the more general construction of a circle to touch the arc and the two bounding radii of any sector in which $\angle AOB < 180°$. Let OA, OB again be the bounding radii and OC the bisector of $\angle AOB$. Let diameters of the circles $A'OA''$, $B'OB''$ be drawn at right angles to BO, AO respectively, A' being on the opposite side of OB to C, A'' on the same; B' on the opposite side of OA to C, B'' on the same. Then CA', CB' cut OB, OA respectively in the points of contact of the inscribed circle with these lines, CA'', CB'' cut OB produced, OA produced respectively in the points of contact of the escribed circle.

(2) Draw the tangent to the arc at C to cut OA produced and OB produced in S and T respectively. Then if distances equal to SC, TC be marked off from S and T respectively along the lines OA and OB respectively in the direction of O, these give points of contact of the inscribed circle, while like distances marked off away from O give those of the escribed circle. This construction is appropriate for any sector where $\angle AOB < 180°$.

P. C. WICKENS.

2082. *Change of axes and rotation centre.*

The formulae for change of axes, given in all books on coordinate geometry, can be looked at in two different ways, of which one, though no more difficult than the other, is for some unknown reason completely neglected.

Suppose $S(x, y) = 0$ to be the equation of an ellipse whose centre is at (p, q) and whose major axis is inclined at an angle θ to the x-axis. The usual procedure is first to change the equation to $S(x+p, y+q) = 0$ and say that this gives the equation of the ellipse referred to axes through its centre parallel to the original axes; and then to use the formulae

$$x = x' \cos\theta - y' \sin\theta, \quad y = x' \sin\theta + y' \cos\theta,$$

to turn the axes through an angle θ and say that

$$S(x\cos\theta - y\sin\theta + p, \quad x\sin\theta + y\cos\theta + q) = 0, \quad \ldots\ldots\ldots\ldots\text{(I)}$$

is the equation of the ellipse referred to its own axes.

If, however, the coordinate axes are *regarded as unchanged throughout*, the equation $S(x+p, y+q)=0$ is that of an ellipse *congruent* to S, and with its axes parallel to those of S, but with its centre at the origin; while the equation (I) is that of an ellipse *congruent to S with its centre at the origin and its axes along the coordinate axes.* This way of looking at the matter has the advantage that the insertion of dashes (done in one of the two cases above) as a precaution against beginners being confused, and the subsequent dropping of dashes to obtain the final result is done away with, since the x and the y refer all the time to the same coordinate axes.

Since the final ellipse at the origin is directly congruent to the original one, the change from one position to the other can be made by a single rotation through an angle θ about the rotation-centre. The coordinates of this point can be found by taking the intersection of the perpendicular bisector of the line joining (p, q) to the origin with the perpendicular bisector of the line joining any other pair of corresponding points, for example, the ends of the major axis. But it can also be found, and more easily, as the point whose coordinates are unchanged by the transformation; that is, by the equations

$$x \cos \theta - y \sin \theta + p = x, \quad x \sin \theta + y \cos \theta + q = y,$$

or

$$x(1-\cos \theta) + y \sin \theta - p = 0, \quad x \sin \theta - y(1-\cos \theta) + q = 0,$$

giving

$$\frac{x}{q \sin \theta - p + p \cos \theta} = \frac{y}{-p \sin \theta - q + q \cos \theta} = \frac{1}{-1 + 2\cos \theta - \cos^2\theta - \sin^2\theta},$$

so that

$$x = \frac{p(1-\cos\theta) - q \sin \theta}{2(1-\cos\theta)}, \quad y = \frac{p \sin \theta + q(1-\cos\theta)}{2(1-\cos\theta)},$$

which, since $\sin\theta/(1-\cos\theta) = \cot \frac{1}{2}\theta$, give finally

$$x = \tfrac{1}{2}(p - q \cot \tfrac{1}{2}\theta), \quad y = \tfrac{1}{2}(p \cot \tfrac{1}{2}\theta + q).$$

In this the conic may equally well be an hyperbola.

If the equation of the conic S is given by the standard general equation, then with the usual notation we have $p = G/C$, $q = F/C$, which slightly alter the above formulae; also θ is given by $\tan 2\theta = 2h/(a-b)$.

Owing to the symmetry of the conic, instead of turning the conic through an angle θ, clockwise, it may be turned through an angle $\pi - \theta$ counter-clockwise and give the same result with the major axis reversed in direction; or this may be described as turning the conic through $\pi + \theta$ clockwise. This is equivalent to changing the signs of both $\cos\theta$ and $\sin\theta$ and gives a second centre of rotation, the equations for which now are

$$x(1+\cos\theta) - y \sin\theta - p = 0, \quad x \sin\theta + y(1+\cos\theta) - q = 0,$$

which give

$$x = \tfrac{1}{2}(p + q \tan \tfrac{1}{2}\theta), \quad y = \tfrac{1}{2}(-p \tan \tfrac{1}{2}\theta + q).$$

As an example, if the major axis of an ellipse runs from (2, 1) to (2, 5) and its minor axis from (1, 3) to (3, 3) so that $\theta = 90°$ and $\tan \frac{1}{2}\theta = \cot \frac{1}{2}\theta = 1$, the two possible rotation centres are $\{\frac{1}{2}(2-3), \frac{1}{2}(2+3)\}$, *i.e.* $(-\frac{1}{2}, \frac{5}{2})$ and

$$\{\tfrac{1}{2}(2+3), \tfrac{1}{2}(-2+3)\}, \textit{ i.e. } (\tfrac{5}{2}, \tfrac{1}{2}).$$

Again, if the major axis runs from (2, 1) to (4, 3) so that $\theta = 45°$ we find a centre of rotation given by the equations

$$x - y + 3\sqrt{2} = x\sqrt{2}, \quad x + y + 2\sqrt{2} = y\sqrt{2}$$

to be the point $\{\frac{1}{2}(1 - 2\sqrt{2}), \frac{1}{2}(5 + 3\sqrt{2})\}$.

If equations without surds are wished for, we can make use of one of the Pythagorean triangles. It is a fairly vigorous exercise to show that by rotation about either $(-6\frac{1}{2}, 5\frac{1}{2})$ or $(\frac{11}{6}, \frac{13}{6})$ the curve

$$145x^2 - 120xy + 180y^2 + 20x - 1560y + 2980 = 0$$

can be brought into coincidence with $x^2/9 + y^2/4 = 1$. C. O. TUCKEY.

2083. *A broken stick.*

Mr. G. A. Bull (Note 2016, *Gazette*, XXXII (1948), pp. 87–8) has considered the problem of finding the probability of breaking a stick into n pieces so that a polygon of n or fewer angles can be formed from them. As he pointed out, in terms of the random division of a finite line this means choosing $(n-1)$ points at random so that each of the n intervals between consecutive points is not greater than half the length of the line. It is also equivalent to choosing n points at random on the circumference of a circle so that they do not all lie on the same semicircle. If the points on the circle are denoted by $X_1, X_2, \ldots, X_n$, we may imagine the arc cut at any one of them, X_n say, so that it becomes a finite line, OX_n say, divided into n intervals by the $(n-1)$ points $X_1, X_2, \ldots, X_{n-1}$. In the case of the circle, each of the n parts into which the arc is divided has an equal probability of being greater than a semicircle; and, similarly, in the case of the finite line, each of the n intervals is equally likely to be greater than half the length of the line. In the latter case, the probability of the first interval OX_1 fulfilling this condition is the probability that all the $(n-1)$ chosen points lie in the second half of the line. This is $1/2^{n-1}$. The probability that any one of the n intervals exceeds half the line is $n/2^{n-1}$, since this can occur in n mutually exclusive ways, and the probability of forming the required polygon is $1 - n/2^{n-1}$.

It is clear that the case of $n=4$ set in C. Smith, *Algebra* (Examples LXI, No. 16), which Mr. Bull mentions, was meant to be tackled by consideration of the analogous problem of points chosen at random on the circumference of a circle. His Ex. No. 14 in the same group states: "If three points are taken at random on a circle, the chance of their lying on the same semicircle is 3/4." This is obviously intended as a guide to the next exercise, No. 15, which is, in effect, the above problem of the stick when $n=3$, the required probability being $1 - \frac{3}{4} = \frac{1}{4}$. Similarly, we may presume that the case of $n=4$ was intended to be solved by analogy with points on a circle.

Treating the problem analytically, we can state that if $(n-1)$ points $X_1, \ldots, X_{n-1}$ are chosen independently at random on the unit line OA, every point of the line being equally likely to be chosen, then the probability that, if the points occur in the given order $X_1, \ldots, X_{n-1}$, the kth interval $X_{k-1}X_k$ exceeds $\frac{1}{2}$ is

$$\prod_{i=1}^{k-2} [Pr\{OX_i \leqslant x\}] \,.\, Pr\{x \leqslant OX_{k-1} \leqslant x + dx\} \,.\, \prod_{i=k}^{n-1} [Pr\{OX_i \geqslant x + \tfrac{1}{2}\}]$$

integrated from $x=0$ to $x=\frac{1}{2}$.

This is $$\int_0^{\frac{1}{2}} x^{k-2} \,.\, \{1 - (x + \tfrac{1}{2})\}^{n-k} \,.\, dx = \frac{1}{2^{n-1}} \int_0^1 y^{k-2} \,.\, (1-y)^{n-k} \,.\, dy,$$

where $y = 2x$,

$$= \{\Gamma(k-1) \,.\, \Gamma(n-k+1)\}/\{\Gamma(n) \,.\, 2^{n-1}\}.$$

It follows that the probability that, whatever the order of the points, the kth interval exceeds $\frac{1}{2}$ is the product of this expression and the number of ways of dividing $(n-1)$ things into three groups containing $(k-2)$, 1, and $(n-k)$ things respectively. This gives

$$\{(n-1) \,.\, {}_{n-2}C_{k-2}\}\{(k-2)!\,(n-k)!\}/\{(n-1)!\,2^{n-1}\} = \tfrac{1}{2}^{n-1},$$

which is independent of k, so that this is the probability that any particular interval exceeds $\frac{1}{2}$. The rest follows immediately.

Now, an alternative method of breaking a stick into n pieces would be to break it at random, take the larger of the two pieces and break it again at random; and continue this process of taking the longer piece each time and breaking it, until there are n pieces in all. What is now the probability of being able to form a polygon of n or fewer angles from these pieces? The only reference that I have seen to this problem is by W. A. Whitworth, who in *DCC Exercises in Choice and Chance* (1897), Ex. 677, poses and gives a solution for the case of $n=3$, obtaining the required probability of forming a triangle as $\frac{1}{3}$. This value is incorrect, as we see by considering the problem analytically.

If the stick OO' is regarded as of unit length and the first break occurs at X_1, so that we may suppose $OX_1 = x_1$ is the shorter piece, and the remaining piece X_1O' is broken at X_2, where $X_1X_2 = x_2 - x_1$ say, then the condition of being able to form a triangle from the pieces OX_1, X_1X_2, X_2O' is violated if $X_2O' > \frac{1}{2}$ or if $X_1X_2 > \frac{1}{2}$, *i.e.* if $x_2 < \frac{1}{2}$ or $x_2 > x_1 + \frac{1}{2}$. The required condition is satisfied if $x_1 + \frac{1}{2} \geqslant x_2 \geqslant \frac{1}{2}$. Now, the probability that X_1 occurs in an interval $(x_1,\ x_1 + dx_1)$, where we have imposed the restriction that $0 \leqslant x_1 \leqslant \frac{1}{2}$, is $dx_1/(\frac{1}{2}) = 2dx_1$; the probability that x_2 then occurs in $(x_2,\ x_2 + dx_2)$ is $dx_2/(1 - x_1)$, and the probability that $x_1 + \frac{1}{2} \geqslant x_2 \geqslant \frac{1}{2}$ for all x_1 lying in $(0, \frac{1}{2})$ is

$$\int_{x_1=0}^{\frac{1}{2}} \int_{x_2=\frac{1}{2}}^{x_1+\frac{1}{2}} 2dx_1 \cdot \frac{dx_2}{1-x_1} = \int_0^{\frac{1}{2}} \frac{2x_1}{1-x_1} \cdot dx_1 = 2\log_e 2 - 1,$$

which is the probability that a triangle can be formed.

In the general case, since after each break the larger piece is selected for rebreaking, the condition for forming a polygon of n or fewer angles is violated only if one of the pieces after the $(n-1)$th break exceeds $\frac{1}{2}$. The probability of this is

$$\int_{x_1=0}^{\frac{1}{2}} \int_{x_2=x_1}^{\frac{1}{2}} \cdots \int_{x_{n-2}=x_{n-3}}^{\frac{1}{2}} \int_{x_{n-1}=x_{n-2}}^{\frac{1}{2}} 2dx_1 \cdot \frac{2dx_2}{1-x_1} \cdot \frac{2dx_3}{1-x_2} \cdots \frac{2dx_{n-1}}{1-x_{n-2}}$$

$$= 2^{n-1} \int_0^{\frac{1}{2}} \int_{x_1}^{\frac{1}{2}} \cdots \int_{x_{n-3}}^{\frac{1}{2}} dx_1 \cdot \frac{dx_2}{1-x_1} \cdots \frac{dx_{n-3}}{1-x_{n-4}} \cdot \frac{1}{1-x_{n-3}} \left\{1 - \frac{1}{2} \cdot \frac{1}{1-x_{n-2}}\right\} dx_{n-2}$$

$$= 2^{n-1} \int_0^{\frac{1}{2}} \int_{x_1}^{\frac{1}{2}} \cdots \int_{x_{n-4}}^{\frac{1}{2}} dx_1 \cdot \frac{dx_2}{1-x_1} \cdots \frac{dx_{n-4}}{1-x_{n-5}} \cdot \frac{1}{1-x_{n-4}} \left\{1 - \frac{1}{2} \cdot \frac{1}{1-x_{n-3}} - \frac{1}{2} \frac{\log_e 2(1-x_{n-3})}{1-x_{n-3}}\right\} dx_{n-3}$$

$$= 2^{n-1} \int_0^{\frac{1}{2}} \left\{1 - \frac{1}{2} \cdot \frac{1}{1-x_1} - \frac{1}{2\,.\,1!} \cdot \frac{\log_e 2(1-x_1)}{1-x_1} - \frac{1}{2\,.\,2!} \cdot \frac{[\log_e 2(1-x_1)]^2}{1-x_1} \cdots - \frac{1}{2(n-3)!} \cdot \frac{[\log_e 2(1-x_1)]^{n-3}}{1-x_1}\right\} dx_1$$

$$= 2^{n-2} \left\{1 - \sum_{r=1}^{n-2} \frac{(\log_e 2)^2}{r!}\right\}.$$

The probability of forming the required polygon is therefore

$$1 - 2^{n-2} \left\{1 - \sum_{r=1}^{n-2} z^r/r!\right\},$$

where $z = \log_e 2$,

$$= 1 - 2^{n-2} \left\{2 - e^z \,.\, \frac{\gamma(n-1, z)}{\Gamma(n-1)}\right\},$$

where $\gamma(n, x)$ is the incomplete gamma function $\int_0^x t^{n-1}e^{-t}dt$

$$=1-2^{n-1}\{1-I(z/\sqrt{n-1},\quad n-2)\},$$

using the notation of Karl Pearson's *Tables of the Incomplete Gamma Function.*

No simple geometrical construction, similar to that given by Mr. Bull for the first method of breaking the stick, appears to be possible for this second method. Whitworth used the same construction for both methods, in the case of $n=3$, and thereby treated points of his figure as representing equally likely cases when, in fact, this was no longer justified. S. RUSHTON.

2084. *Triangle with integral sides and integral medians.*

After reading Note 1421 I felt that readers might be interested in my own discoveries in this matter.

The following formula due to Euler gives triangles with integral sides and integral medians :

$$a=p(m+n)-q(m-n),\quad b=p(m-n)+q(m+n),\quad c=2(mp-nq),$$

where $$p=(m^2+n^2)(9m^2-n^2),\quad q=2mn(9m^2+n^2).$$

Taking $m=1$ and $n=2$, we get the sides

$$a=131,\quad b=127,\quad c=158.$$

Double these sides and we get

$$a=262,\quad b=254,\quad c=316,$$

and the medians of this triangle are 255, 261 and 204.

Now here is the discovery. If we take $\frac{2}{3}$ of each of these medians we get 170, 174, 136, and these numbers are the sides of a triangle with medians 131, 127 and 158.

I think these numbers are the smallest possible ; I think it is impossible to find a triangle with integral sides, integral medians and integral area.

J. TRAVERS.

2085. *An appeal to authors.*

Let me offer, with all deference, two suggestions that affect mathematical works in general. Has not the time come to call a halt to the meticulous triple reference markings that pervade so many books? For the reader of an interesting mathematical argument it is a little distracting to be told a dozen times on one page that he is reading Section Three of Chapter Five, especially when the post-war conditions encourage closely packed printing. Also it takes appreciably longer to find the reference 4 . 3 . 5 as required on p. 189 than it would if the direction were explicitly made to p. 107. And again, is it entirely necessary to use the asterisk as an essential part of mathematical notation? An expression such as $a_0{}^*b_2{}^*-2a_1{}^*b_1{}^*+a_2{}^*b_0{}^*$ admittedly holds the eye ; but the eye is caught by the galaxy of stars, like so many signal lights, away from the algebraic structure of the form. Or, to change the metaphor, the asterisk is like a red rag to the present writer, if only because his eyesight is astigmatic! Perhaps others are affected consciously or unconsciously in the same way? H. W. T.

2086. *Reduction of a square matrix by the operator* $T^{-1}(\ )T$.

I. If A is a square matrix and $B=T^{-1}AT$, T can be chosen so that B is diagonal of the form $(\lambda_p\delta_{pq})$ where the λ_p are the latent roots of A, *provided that A has a complete set of independent proper vectors.*

In fact, (*a*) : If x is a proper vector of A, so that $Ax=\lambda x$, then

$$BT^{-1}x=T^{-1}Ax=\lambda T^{-1}x,$$

so that $T^{-1}x$ is a proper vector of B corresponding to the same root. And (*b*) : the p-th column-vector of T is Te_p, where e_p is the p-th basic vector, having unity as its p-th component and zero for all the others.

Hence if we take the set of proper vectors of A as the column-vectors of T, B will have the basic vectors as proper vectors, with same roots as A ; which proves the proposition.

II. If the latent roots of A are distinct, the required condition is satisfied ; otherwise it may or may not be. But it is always satisfied if A is *Hermitian, skew-Hermitian, or unitary*. Further, *the proper vectors are mutually perpendicular* (in the sense that $\bar{x}'y=0$), so that T can be a unitary matrix ($\bar{T}'T=I$). We need two lemmas : (i) Given any vector x (" normalised " so that $\bar{x}'x=1$), we can form in an infinity of ways a unitary matrix having x as a column-vector. This is easily proved. (ii) If T is unitary and A belongs to one of our three classes, B belongs to the same class. For we find

$$\bar{B}'=\bar{T}'\bar{A}'T \quad \text{and} \quad \bar{B}'B=\bar{T}'\bar{A}'AT,$$

so that if $\bar{A}'=\pm A$, or $\bar{A}'A=I$, B has the same property.

Now if A has a root λ_1 (single or multiple), it has a corresponding proper vector. If this is normalised and taken as the first column-vector of T, B has e_1 as a proper vector, hence its first column will be λ_1 followed by zeros. Since B, like A, belongs to one of our three classes, it is easily seen that its first *row* will also have zeros after its first term. The other roots will be roots of the matrix got by striking out the first column and row of B, and any one of them will give a proper vector which is a linear combination of $e_2, e_3 \ldots e_n$. Hence the corresponding proper vector of A will be a linear combination of the second, third, ... , n-th column-vectors of T, and therefore perpendicular to the first. If this be taken as the second column of T, the second column and row of B will have all their terms zero except the second. which will be λ_2. Proceeding in this way, the result follows. M. F. EGAN.

2087. *Note on spherical geometry.*

LEMMA : If L is the middle point of the side BC of the spherical triangle ABC, each of the conditions

$$\text{(i) } b+c=\pi, \quad \text{(ii) } B+C=\pi, \quad \text{(iii) } AL=\tfrac{1}{2}\pi$$

implies the other two.

For if AB and AC meet again at A', each of our three conditions means that the triangles ABC, ALC are identically equal to $A'CB$, $A'LB$ respectively.

I. *Triangle covering a quarter of the sphere.*

Let L, M, N be the middle points of the sides of a triangle. If $AL+LB=\pi$, or, what is equivalent by our lemma, if $LN=\frac{1}{2}\pi$, then the angle $LAB=\pi-B$, and similarly $LAC=\pi-C$, so that $A+B+C=2\pi$. Conversely, if

$$A+B+C=2\pi,$$

take the point L on BC such that $LAB=\pi-B$, and consequently $LAC=\pi-C$. We get

$$LB=\pi-LA=LC\,; \quad \text{also } LN=LM=\tfrac{1}{2}\pi.$$

Hence :

If one median is the supplement of half the corresponding side, or if the distance between the middle points of one pair of sides is a quadrant, the same relations hold however we choose the sides, and the triangle covers a quarter of the sphere ; and conversely.

II. In general: *Given two vertices A, B and the area of the triangle, the locus of C is a circle, and the middle points of AC and BC lie on the great circle parallel to this circle.*

For if O is the pole of the circumcircle of the colunar triangle ABC', OA and OB make each the angle $\pi - S$ with AB. Since S is given, O is fixed and so is the circle locus of C'. Since C is diametrically opposite to C', its locus is a circle equal and parallel to the circle (C'), and having O as its (remote) pole. Also, applying our lemma to the triangles COA, COB, we have

$$CO + OA = CO + OB = CO + OC' = \pi,$$

hence the distances from O of the middle points of AC and BC are each a quadrant.

I do not suppose there is anything new in these results, but our lemma brings them out very simply, without any trigonometry. M. F. EGAN.

2088. *Approximation to $\sqrt{x}$.*

Two well-known ways of approximating to the square root of a number x consist in choosing a number a, conveniently but not necessarily the integer next below $\sqrt{x}$, and

(1) writing
$$a_1 = \tfrac{1}{2}(a + x/a),$$
$$a_2 = \tfrac{1}{2}(a_1 + x/a_1),$$

and so on, whence $a_n \rightarrow \sqrt{x}$;
or (2) putting
$$x = a^2 + b$$

and forming the successive convergents p_1/q_1, p_2/q_2 ... of the continued fraction

$$a + \frac{b}{2a+}\,\frac{b}{2a+}\,\ldots .$$

Thus if we take $x = 2$, $a = 1$, we obtain

$$a_1 = 3/2, \quad a_2 = 17/12, \quad a_3 = 577/408, \ldots,$$

and we find that these are equal respectively to

$$p_2/q_2, \quad p_4/q_4, \quad p_8/q_8.$$

This suggests that we have generally

$$a_n = p_m/q_m, \quad \text{where } m = 2^n.$$

To prove this, we have

$$p_{n+1} = 2ap_n + bp_{n-1},$$
$$q_{n+1} = 2aq_n + bq_{n-1}.$$

Solving these difference equations by the ordinary rule, and noting that $p_1 = a$, $q_1 = 1$, $p_2 = 2a^2 + b$, $q_2 = 2a$, we obtain

$$p_n/q_n = \sqrt{(a^2+b)}\,.\,(\alpha^n + \beta^n)/(\alpha^n - \beta^n),$$

where α, β are the roots of $x^2 - 2ax - b = 0$, whence it is easy to see that

$$p_n/q_n + (a^2+b)q_n/p_n = 2p_{2n}/q_{2n}.$$

Thus the $2n$th convergent of the continued fraction is obtained from the nth by the same process as that by which we derive a_{n+1} from a_n. Since $a_1 = p_2/q_2$, the desired result follows.

It may be mentioned that there is some reason to suppose that method (1) was known to the Babylonians, while the series of convergents to $\sqrt{2}$ given by method (2) is found in Plato. H. W. CHAPMAN.

2089. *A chain rule for use with determinants and permutations.*

This rule was described by N. and W. J. Chater in Vol. 31, pp. 279–87, and may be stated as follows : In the expansion of the determinant having a_s^r in row r and column s, the sign of the term $a_{s_1}^1 a_{s_2}^2 \dots a_{s_n}^n$ is + or − according as the number of cycles in the expression of the permutation $(s_1 s_2 \dots s_n)$ differs from n by an even or odd number.

In a recent issue of *Mathematical Reviews* (Vol. 9, p. 323), J. Riordan has pointed out that this rule is proved very simply as follows : If the permutation has c_k cycles of order k, then

$$c_1 + 2c_2 + \dots + nc_n = n,$$

and therefore

$$n - (c_1 + c_2 + \dots + c_n) = c_2 + 2c_3 + \dots + (n-1)c_n.$$

Since a k-cycle is equivalent to $k-1$ transpositions, the Chater rule is therefore identical with the usual one on the evenness or oddness of the permutation. D. PEDOE.

2090. $1 = 0$.

As a warning to the unwary, the following may have some interest : by integration by parts,

$$\int \frac{f'(x)}{f(x)}\,dx = \frac{f(x)}{f(x)} + \int \frac{f(x)\,.\,f'(x)}{\{f(x)\}^2}\,dx = 1 + \int \frac{f'(x)}{f(x)}\,dx.$$

Hence $1 = 0.$ H. V. L.

2091. *The digits in the decimal form of π.*

In Note 2004, Vol. XXXII, February 1948, the first 810 decimals of π calculated by Mr. D. F. Ferguson are given. In this note it was stated that the figures do not agree with those given by Shanks after the first 527 decimals.

In the *Zeitschrift fur den mathematischen und naturwissenschaftlichen Unterricht*, 1941, p. 124, the relative frequency (*relative Häufigkeit*) of the digits 0, 1, 2, ..., 9 in the first 700 decimals given by Shanks is exhibited. Each of them may be expected to appear about 70 times, but the actual frequencies are

0	1	2	3	4	5	6	7	8	9
73	77	74	72	71	63	70	51	71	78

It is surprising that the digit 7 diverges further from the mean value than the others, whose divergence is about 10 per cent.

But with Mr. Ferguson's calculation, the digit 7 does not exhibit this property. We have the following table :

	0	1	2	3	4	5	6	7	8	9	
First 500 decimals	45	59	54	50	53	50	48	36	53	52	Shanks and Ferguson
First 700 decimals	64	76	75	71	73	67	70	65	68	71	Ferguson
First 700 decimals	73	77	74	72	71	63	70	51	71	78	Shanks
First 800 decimals	74	92	83	79	80	73	77	75	76	91	Ferguson

The digit seven is therefore not different from the others. Its apparently irregular conduct is not due to a new mystic property of the figure 7, but goes back to an error in the calculation and disappears with the correction of this error. F. BUKOVSZKY.

[De Morgan, *Budget of Paradoxes* (second ed., II, p. 65), had noted that in the first 608 figures given by Shanks, 7 occurs only 44 times, whereas the mean value is 61. He remarks : " Here is a field of speculation in which two branches of inquirers might unite. There is but one number which is treated

with an unfairness which is incredible as an accident; and that number is the mystic number *seven*! If the cyclometers and the apocalyptics would lay their heads together until they come to an unanimous verdict on this phenomenon, and would publish nothing until they are of one mind, they would earn the gratitude of their race."]

2092. *On Note* 2028: *Divergence of harmonic series.*

The neat argument

$$s_n = 1 + 2^{-s} + \ldots + n^{-s} > n \,.\, n^{-s} = n^{1-s} \to \infty ,$$

to establish the divergence of Σn^{-s} when $s < 1$, implies a challenge to try the same method for the harmonic series Σn^{-1}. Direct application of the method (giving only $s_n > 1$) is out of the question. But would the following argument help to bridge the gap that some students (or teachers as guardians of their students' minds) feel exists between the simplicity of this method and the relative artificiality of the usual grouping?

Let $$s(n, p) = (n+1)^{-1} + (n+2)^{-1} + \ldots + (n+p)^{-1},$$

so that, as above, $$s(n, p) \geqslant p \,.\, (n+p)^{-1}.$$

Hence, if k is any positive number less than 1, *e.g.* $k = \frac{1}{2}$, then $s(n, p) \geqslant k$ if $p \geqslant kn + kp$, *i.e.* $p \geqslant nk/(1-k)$, *e.g.*:

$$s(n, p) \geqslant \tfrac{1}{2} \text{ if } p \geqslant n.$$

Thus, taking $k = \frac{1}{2}$, at whatever stage (n) we stop in the series, we can add at least $\frac{1}{2}$ to the sum to that stage by doubling the number of terms; and, by repeating this process, we can have as many $\frac{1}{2}$'s as we like; and the divergence is proved. This is, of course, equivalent to the usual groupings.

Returning to examine more closely the direct consideration of s_n (when $s = 1$), let A, G, H, L be the arithmetic mean, the geometric mean, the harmonic mean and the least of the n terms of s_n, so that

$$A = s_n/n, \quad G = (n!)^{-1/n}, \quad H = 2/(n+1), \quad L = 1/n$$

and $$L \leqslant H \leqslant G \leqslant A.$$

The original argument, which uses $A \geqslant L$, leads to $s_n \geqslant 1$, which is not sufficient to prove $s \to \infty$. Will comparison of A with H or with G, instead of with L, improve the result sufficiently to give the desired proof? The answer is: No. The relation $A \geqslant H$ leads only to

$$s_n \geqslant 2n/(n+1) = 2 - 2/(n+1),$$

i.e. merely $s_n \geqslant 2$, still short of $s_n \to \infty$; and $A \geqslant G$ leads to $s_n \geqslant n(n!)^{-1/n}$, which (by Stirling's theorem!) tends to the limit e, giving something short of $s_n \geqslant e$, —again not $s_n \to \infty$. Plainly the simple proof sought is not attainable this way. Almost equally plainly the relatively crude means will have to be replaced by something much closer to the actual value of s_n. The use of integration to obtain a close approximate formula for the sum s_n is the most natural next step to try; and we have arrived at the comparison with the integral of x^{-1}, which is the second of the two methods we set out to try to avoid.

But is there any real reason why these arguments of grouping and integration should be shunned? C. WALMSLEY.

2093. *On skew-symmetrical determinants.*

The theorem that *a skew-symmetrical determinant of odd order is zero* is one of the easiest theorems in the theory of determinants, but its fellow:

A skew-symmetrical determinant of even order is the square of a polynomial function of its elements.

is one which the first-year undergraduate finds difficult. I give here a proof of this theorem which my own students appear to prefer to the one usually given in textbooks. It begins and ends in the usual way, but the middle section is different.

Let
$$\Delta_1 \equiv \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$
be a skew-symmetrical determinant of *odd order n*, so that
$$a_{rs} = -a_{sr}, \quad a_{rr} = 0, \quad \Delta_1 = 0.$$

Let A_{rs} denote the cofactor of a_{rs} in Δ_1. Then since each column of A_{sr} is (-1) times the corresponding row in A_{rs}, we have
$$A_{sr} = (-1)^{n-1} A_{rs} = A_{rs}. \quad \dots\dots (1)$$

Moreover, since $\Delta_1 = 0$, the cofactors of any row of Δ_1 are proportional to those of any other row, so comparing those of the first row with those of the rth we get
$$A_{11}/A_{r1} = A_{1r}/A_{rr},$$
and this gives, by (1),
$$A_{1r} = (A_{11}A_{rr})^{\frac{1}{2}}, \quad \dots\dots (2)$$
where we assume that the correct sign is attached to the R.H.S.

Now let
$$\Delta \equiv \begin{vmatrix} a_{11} & \dots & a_{1n} & x_1 \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} & x_n \\ -x_1 & \dots & -x_n & 0 \end{vmatrix}.$$

Then Δ is a skew-symmetrical determinant of even order. Using row-into-row multiplication together with the properties of cofactors, we have
$$A_{11}\Delta = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix} \times \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & x_1 \\ a_{21} & a_{22} & \dots & a_{2n} & x_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & x_n \\ -x_1 & -x_2 & \dots & -x_n & 0 \end{vmatrix}$$
$$= \begin{vmatrix} \Delta_1 & 0 & \dots & 0 & -\sum\limits_{r=1}^{n} A_{1r}x_r \\ a_{12} & a_{22} & \dots & a_{n2} & -x_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} & -x_n \\ x_1 & x_2 & \dots & x_n & 0 \end{vmatrix}.$$

Since $\Delta_1 = 0$ and $(n+1)$ is even, this gives
$$A_{11}\Delta = \left(\sum_{r=1}^{n} A_{1r}x_r\right) \begin{vmatrix} a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \\ x_1 & x_2 & \dots & x_n \end{vmatrix}$$

$$=(\sum_{r=1}^{n} A_{1r}x_r)(\sum_{r=1}^{n} A_{r1}x_r)$$

$$=(\sum_{r=1}^{n} A_{1r}x_r)^2$$

by (1). Using (2) we get

$$\Delta=\{\sum_{r=1}^{n} (A_{rr})^{\frac{1}{2}}x_r\}^2. \qquad (3)$$

Now $A_{11}, \dots, A_{nn}$ are skew-symmetrical determinants of the even order $n-1$. If each is a square of a polynomial function of its elements, then, by (3), the same is true for Δ. Hence the theorem is true for determinants of order $n+1$ if it is true for ones of order $n-1$. But

$$\begin{vmatrix} 0 & x \\ -x & 0 \end{vmatrix}=x^2,$$

so the theorem is true for determinants of order 2, and therefore, by induction, it is true for determinants of any even order. IDA W. BUSBRIDGE.

2094. *On Note* 2023.*

1. The subject of Note 2023 is a non-inductive proof of the formula

$$\frac{c_1}{1}-\frac{c_2}{2}+\dots+(-1)^{n-1}\frac{c_n}{n}=1+\frac{1}{2}+\dots+\frac{1}{n}, \qquad (1)$$

where c_r denotes the binomial coefficient ${}^nC_r$. This can be proved more briefly as follows.

By means of the substitution $x=1-t$, we see that

$$\int_0^1 \frac{1-x^n}{1-x}\,dx=\int_0^1 \{1-(1-t)^n\}\frac{dt}{t},$$

and it follows that

$$\int_0^1 (1+x+x^2+\dots+x^{n-1})dx=\int_0^1 \{c_1-c_2t+c_3t^2-\dots+(-1)^{n-1}c_nt^{n-1}\}dt.$$

Integrating, we have (1).

2. This method can be used to obtain generalisations of (1). For example, starting with the equation

$$\int_0^1 \frac{x^a(1-x^n)}{1-x}\,dx=\int_0^1 (1-t)^a\{1-(1-t)^n\}\frac{dt}{t} \quad (a>-1),$$

we obtain the formula

$$\frac{1}{a+1}+\frac{1}{a+2}+\dots+\frac{1}{a+n}=\sum_{r=1}^{n}(-1)^{r-1}\frac{(r-1)!\,c_r}{(a+1)(a+2)\dots(a+r)}.$$

If we start with

$$\int_0^1 (1-x)^{a-1}(1-x^n)dx=\int_0^1 t^{a-1}\{1-(1-t)^n\}dt=\int_0^1 t^{a-1}dt-\int_0^1 t^{a-1}(1-t)^n dt,$$

* [Several contributors have remarked that while Mr. Parameswaran's direct proof is most ingenious, the average Sixth-Form mathematician might be expected, not unreasonably, to hit on the line of argument given by Dr. Busbridge in § 1.—Ed.]

in which the first two are equal when $a > -1$ and the second two when $a > 0$, we obtain

$$\sum_{r=1}^{n} \frac{(r-1)!}{(a+1)(a+2)\dots(a+r)} = \sum_{r=1}^{n} \frac{(-1)^{r-1} c_r}{a+r} = \frac{1}{a} - \frac{n!}{a(a+1)\dots(a+n)}.$$

The last two of these form Mr. Parameswaran's starting point.

3. Formulae which are not generalisations of (1) can be obtained by repeated integration. For example,

$$\int_0^1 \frac{dy}{y} \int_0^y \frac{1-x^n}{1-x}\, dx = \int_0^1 \frac{dy}{y} \int_{1-y}^1 \{1-(1-t)^n\} \frac{dt}{t},$$

and this gives the formula

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} = \sum_{r=1}^{n} (-1)^{r-1} \frac{c_r}{r} \left(1 + \frac{1}{2} + \dots + \frac{1}{r}\right).$$

IDA W. BUSBRIDGE.

2095. *A query.*

What is known, or what can be said, about the differential equations

$$\vartheta^n y = y, \quad \vartheta'^n y = y,$$

where $$\vartheta \equiv \frac{d}{dx} \log, \quad \vartheta' = \log \frac{d}{dx}.$$

G. E. CRAWFORD.

2096. *Centre of pressure.*

1. The relation between C.G. and C.P. explained in Note 1832 (*Gazette*, July 1945) has a somewhat wider application ; the following table gives some examples :

Body.	C.G.	C.P.
Circle, just immersed - - - - - - -	$1\frac{1}{2}/3$	$2\frac{1}{2}/4$
Parabola, bounded by double ordinate ; axis vertical, vertex in surface - - - - - - - -	$1\frac{1}{2}/2\frac{1}{2}$	$2\frac{1}{2}/3\frac{1}{2}$
Same parabola, just immersed, axis horizontal -	2/4	3/5
Area bounded by parabola (axis in surface) tangent at vertex and horizontal line - - - -	3/4	4/5

The first two examples will have prepared the reader for the case of a semicircular area with the diameter in the surface ; expressing C.G. as

$$\left(\frac{64-12\pi}{9\pi^2-64}\right) \Big/ \left(\frac{48\pi-9\pi^2}{9\pi^2-64}\right), \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(A)$$

the value of C.P. is obtained when both numerator and denominator are increased by one.

2. Taking as axes OX vertically downwards and OY in the surface, let the area " under " the curve $y = f(x)$ from $x = 0$ to $x = 1$ be immersed in homogeneous fluid ; atmospheric pressure being neglected. Two general results can be stated :

(i) If $f(x)$ is of the form $x^r(1-x)^s$ where $r, s > -1$, then

$$\text{C.G.} = \frac{r+1}{r+s+2}, \quad \text{C.P.} = \frac{r+2}{r+s+3}. \quad \dots\dots\dots\dots\dots\dots\dots(B)$$

This depends on the properties of Beta functions. The examples in the table

correspond to the values $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0)$, $(1, 1)$ and $(2, 0)$ of r and s respectively, the examples in Note 1832 to the values $(0, 0)$, $(1, 0)$, and $(0, 1)$; (the expression (A) above is not of this type). It will be seen that if $r+1$; $s+1$ are integers prime to each other, the expression (B) for C.G. is in its simplest form, and the rule is applicable without " cheating ".

(ii) If it is known that the rule applies to a curve $y=f(x)$, then it also applies to $y=f(1-x)$ if the expressions used for C.G. have the same denominators; this is easily proved. H. M. FINUCAN.

2097. *Discriminants and resolvents.*

The familiar definition of the discriminant of an equation involving a single unknown as the *simplest* function of the coefficients, in a rational and integral form, whose vanishing expresses the condition for equal roots, is in direct opposition to the accepted usage of discriminants in the theory of differential equations. It is well known that the multiplicity of the factors of the p-discriminant of a differential equation $\phi(x, y, p)=0$ helps to determine the character of the loci given by the vanishing of these factors, but if the p-discriminant is taken by definition to be the simplest expression whose vanishing expresses the condition for equal roots, then clearly each factor must be taken only to the first degree and the discriminant loses its power of discrimination.

On the other hand, to define the discriminant of the equation $p(x)=0$, where $p(x)$ is a polynomial of the nth degree, as the resolvent of the pair of equations $p(x)=0$, $p'(x)=0$, a definition that is also frequently adopted, introduces a superfluous factor into the discriminant. For if $x=\alpha_r$, $1 \leqslant r \leqslant n$, are the roots of the equation $p(x)=0$, and if the leading coefficient of $p(x)$ is a_0, then the *resolvent* of the equations $p(x)=0$, $p'(x)=0$ is

$$a_0^{n-1}p'(\alpha_1) \cdot p'(\alpha_2) \cdot \ldots \cdot p'(\alpha_n)=(-1)^{\frac{1}{2}n(n-1)}a_0^{2n-1}[1\alpha_2\alpha_3^2\alpha_4^3 \ldots \alpha_n^{n-1}]^2.$$

Since the product of the squared differences of the roots

$$[1\alpha_2\alpha_3^2 \ldots \alpha_n^{n-1}]^2=\Delta^2$$

is a symmetric function of order $2(n-1)$, therefore $a_0^{2n-2}\Delta^2$ is a rational integral function of the coefficients whose vanishing is a necessary and sufficient condition for the equation $p(x)=0$ to have at least two equal roots, and so $a_0^{2n-2}\Delta^2$ is the discriminant of the equation.

It follows that the ratio of the resolvent of $p(x)=0$, $p'(x)=0$ to the discriminant of $p(x)=0$ is $\pm a_0$.

The presence of this superfluous factor in the resolvent shows that the discriminant cannot be obtained by elimination; the discriminant must therefore be defined explicitly in the form

$$\text{(i)} \quad a_0^{2n-2}\Delta^2,$$

or in one of the equivalent forms:

$$\text{(ii)} \quad a_0^{n-2}p'(\alpha_1) \cdot p'(\alpha_2) \cdot \ldots \cdot p'(\alpha_n),$$

$$\text{(iii)} \quad a_0^{n-1}p(\beta_1) \cdot p(\beta_2) \cdot \ldots \cdot p(\beta_{n-1}),$$

where $x=\alpha_r$, $1 \leqslant r \leqslant n$ are the roots of $p(x)=0$ and $x=\beta_r$, $1 \leqslant r \leqslant n-1$, are the roots of $p'(x)=0$.

Example. To find the singular solutions of the differential equation

$$\phi(p) \equiv 8p^3x-12p^2y+9y=0.$$

Since $\phi_p \equiv 24p(px-y)$, the roots of $\phi_p=0$ are $p=0$, $p=y/x$, and so the discriminant of $\phi(p)=0$, by (iii), is

$$x^2\phi(0)\phi(y/x)=9y^2(9x^2-4y^2).$$

The multiplicity of the factor y shows that $y=0$ is both an envelope and a cusp locus ; the lines $2y \pm 3x=0$ are envelopes.

The *simplest* rational integral function of the coefficients of $\phi(p)$ whose vanishing expresses the condition for equal roots is, of course, $y(9x^2-4y^2)$, and this expression does not reveal the dual role of the locus $y=0$.

On the other hand, the resolvent of $\phi=0$, $\phi_p=0$ is

$$x^3\phi(0)\phi(y/x)=9xy^2(9x^2-4y^2),$$

and the additional root $x=0$ is not a singular solution of the differential equation. R. L. G.

2098. *Genius in disguise.*

During the war a friend of mine, in charge of a group of natives in Central Africa, was responsible for seeing that they understood the mechanism of a gun of some kind. The teaching was in the hands of a capable sergeant and, though the natives had been taken straight from the fields, he managed to teach them addition and subtraction with numbers of one digit. He could not, however, get one man to understand that $4-4=0$. At last a brilliant idea came to him. There were some fairly heavy boxes in the room, so he ordered the native to drag four of them before the class. " Now," said he, " if I tell you to take those four boxes outside, how many will be left? " There was an uprush of feeling in him, pride at his pedagogic skill, elation at his surmounting of the difficulty, as he saw the light of intelligence dawn on the native's face. But alas! " Two," said the native. " How the ' sanguinated aspirated labial ' do you get that? " spluttered out the exasperated sergeant. In quiet amazement at the unexpected outburst the man slowly replied, " I can only carry two." R. S. WILLIAMSON.

2099. *The Euler-Savary Formula.*

Cartan's notation in the use of the method of moving axes is so clear and economical that, by way of advocating its wider adoption, we shall use it in the following proof of the Euler-Savary formula for the radius of curvature of the envelope of a curve drawn in a lamina which has a prescribed motion.

If Ax_1, Ax_2 are perpendicular axes drawn in a lamina L_1, and $P(x_1, x_2)$ is a fixed point in it, an infinitely small displacement of the lamina is summed up in the formulae :

$$\begin{aligned}
d\vec{P}&=d\vec{A}+x_1 d\vec{e}_1+x_2 d\vec{e}_2,\\
d\vec{A}&=\omega_1\vec{e}_1+\omega_2\vec{e}_2,\\
d\vec{e}_1&=\omega_{12}\vec{e}_2,\\
d\vec{e}_2&=-\omega_{12}\vec{e}_1,
\end{aligned}$$

where $\vec{e}_1\,\vec{e}_2$ are co-ordinate vectors along Ax_1Ax_2 and ω_1, ω_2, ω_{12} are differential forms in the parameters which specify the position of the axes. These differential forms may, in this the simplest case, be obtained at once by elementary methods. Thus if A has co-ordinates $(\lambda_1\lambda_2)$ relative to fixed axes $O\xi_1, O\xi_2$, and if θ is the angle which Ax_1 makes with $O\xi_1$, then

$$\begin{aligned}
\omega_1&=\cos\theta d\lambda_1+\sin\theta d\lambda_2,\\
\omega_2&=-\sin\theta d\lambda_1+\cos\theta d\lambda_2,\\
\omega_{12}&=d\theta.
\end{aligned}$$

If Ax_1 is the positive half-tangent to a curve at a point A on it, *i.e.* if the frame Ax_1x_2 is the " natural frame associated with the curve ", the formulae (the Frenet formulae) read :

$$\overrightarrow{dA} = ds \,.\, \overrightarrow{e_1},$$

$$\overrightarrow{de_1} = \kappa\, ds \,.\, \overrightarrow{e_2},$$

$$\overrightarrow{de_2} = -\kappa\, ds \,.\, \overrightarrow{e_1},$$

where κ is the curvature of the curve at A.

If the lamina Ax_1x_2 is carried by another lamina L_2 which is itself moving, the differential forms depend on the parameters which specify the position of L_1 relative to L_2 and those specifying the position of L_2. If the latter are kept constant, the formulae relate to the relative motion.

In the problem which confronts us, let Ax_1x_2 be the natural frame at the point of contact of the curve C with its envelope E. It is also the natural frame at the point A of the envelope. Relatively to the lamina, the frame Ax_1x_2 moves with an instantaneous motion specified by

$$d_1\overrightarrow{A} = ds \,.\, \overrightarrow{e_1},$$

$$d_1\overrightarrow{e_1} = \kappa\, ds \,.\, \overrightarrow{e_2},$$

$$d_1\overrightarrow{e_2} = -\kappa\, ds \,.\, \overrightarrow{e_1},$$

where κ is the curvature of C and ds its element of arc. The instantaneous motion of the lamina itself is specified by

$$d_2\overrightarrow{A} = \omega\, dt \,.\, R \,.\, \overrightarrow{IA},$$

$$d_2\overrightarrow{e_1} = \omega\, dt \,.\, \overrightarrow{e_2},$$

$$d_2\overrightarrow{e_2} = -\omega\, dt \,.\, \overrightarrow{e_1},$$

where R is the rotation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and I the instantaneous centre and ω the angular velocity of the lamina.

The result of compounding these two motions is to give the motion of the natural frame along the envelope. It follows that

$$\left.\begin{aligned} \overrightarrow{dA} &= ds\overrightarrow{e_1} + \omega\, dtR \,.\, \overrightarrow{IA} = dS \,.\, \overrightarrow{e_1} \\ \overrightarrow{de_1} &= (\kappa\, ds + \omega\, dt)\overrightarrow{e_2} = K \,.\, dS \,.\, \overrightarrow{e_2} \\ \overrightarrow{de_2} &= -(\kappa\, ds + \omega\, dt)\overrightarrow{e_1} = -K\, dS\overrightarrow{e_1} \end{aligned}\right\}, \quad \text{.....................(1)}$$

where K is the curvature of the envelope at the point A and dS its element of arc.

We infer

(i) $R \,.\, \overrightarrow{IA}$ has the same direction as $\overrightarrow{e_1}$, *i.e.* the scalar product $\overrightarrow{IA} \,.\, \overrightarrow{e_1} = 0$: the instantaneous centre lies on the normal at A, a particular case of the " theorem of three centres ".

(ii) $dS = ds - IA \,.\, \omega\, dt$.

(iii) K is given by

$$K(ds - IA \,.\, \omega\, dt) = \kappa\, ds + \omega\, dt. \quad \text{..............................(2)}$$

The relation $\vec{IA} \cdot \vec{e}_1 = 0$ holds for all positions of A and serves to determine the relation between s and t. Differentiation gives

$$(\vec{dA} - \vec{dI}) \cdot \vec{e}_1 + \vec{IA} \cdot \vec{de}_1 = 0.$$

The vector $\vec{dI}$ is of magnitude $d\sigma$ and directed along the tangent at I to the space centrode, $d\sigma$ being the element of arc. If IA is inclined to this tangent at an angle θ, substituting for $\vec{dA}$ from (1) and writing $V = \frac{d\sigma}{dt}$ gives

$$ds - V \sin\theta \, dt + IA \cdot \kappa \, ds = 0. \quad \text{.............................(3)}$$

Substituting in (2) and writing $K = \frac{1}{\rho_2}$, $\kappa = 1/\rho_1$,

$$\frac{\frac{1}{\rho_2} - \frac{1}{\rho_1}}{1 + \frac{IA}{\rho_1}} = \frac{\omega\left(1 + \frac{IA}{\rho_2}\right)}{V \sin\theta},$$

i.e.

$$\frac{\rho_1 - \rho_2}{IC_1 \cdot IC_2} = \frac{\omega}{V \sin\theta},$$

where C_1 and C_2 are the centres of curvature of C and E respectively. This is the Euler-Savary formula. Writing $\rho_1 = 0$ the formula gives the radius of curvature of a point roulette described by a point P,

$$\frac{\rho_2}{IP \cdot IC_2} = \frac{-\omega}{V \sin\theta},$$

and if the point P is at a point of inflexion of its path, the ratio $\frac{\rho_2}{IC_2}$ has the value unity, giving the equation of the circle of inflexions :

$$\frac{1}{IP} = \frac{1}{r} = \frac{-\omega}{V \sin\theta}.$$

Writing $\frac{1}{\rho_1} = 0$, the formula gives the curvature of a line roulette,

$$\frac{1}{IC_2} = \frac{\omega}{V \sin\theta}.$$

R. BUCKLEY AND E. V. WHITFIELD.

2100. *To construct a symmetrical, pandiagonal magic cube of oddly even order* $2n \geqslant 10$.

Fill up each octant with a cube of order n having the desired properties. Then add to each of its elements n^3 times the corresponding elements of a symmetrical, pandiagonal cube formed from the numbers 0 to 7 each repeated n^3 times, the eight numbers in eight cells with coordinates all congruent mod n being all different.

2. For the cells with z coordinate even $(=2t)$, use the numbers 0, 3, 5, 6. Then the other alternate layers can be filled up by symmetry with 7, 4, 2, 1.

3. In a set of four cells with coordinates

$$\begin{array}{ll} (2r,\ 2s,\ 2t), & (2r+n,\ 2s,\ 2t), \\ (2r,\ 2s+n,\ 2t), & (2r+n,\ 2s+n,\ 2t), \end{array}$$

enter one of the following n sets of four numbers :

(1) (for n prime to 3)

$$\begin{vmatrix} 3 & 0 \\ 6 & 5 \end{vmatrix} \begin{vmatrix} 6 & 3 \\ 0 & 5 \end{vmatrix} \begin{vmatrix} 6 & 3 \\ 0 & 5 \end{vmatrix};$$

with $(n-3)/2$ repetitions of $\begin{vmatrix} 3 & 0 \\ 6 & 5 \end{vmatrix}$ and $\begin{vmatrix} 5 & 6 \\ 0 & 3 \end{vmatrix}$

or (2) (for n divisible by 3)

3 repetitions of $\begin{vmatrix} 3 & 0 \\ 6 & 5 \end{vmatrix}$, six of $\begin{vmatrix} 6 & 3 \\ 0 & 5 \end{vmatrix}$

and $(n-9)/2$ of $\begin{vmatrix} 3 & 0 \\ 6 & 5 \end{vmatrix}$ and $\begin{vmatrix} 5 & 6 \\ 0 & 3 \end{vmatrix}$.

These are to be associated with the n different residues $0, 1, 2, \ldots n-1$ of $r+s+t$ mod n.

4. Corresponding elements of the sets (1) total $\begin{vmatrix} 4n+3 & 3n-3 \\ 3n-3 & 4n+3 \end{vmatrix}$,

while those of (2) total $\begin{vmatrix} 4n+9 & 3n-9 \\ 3n-9 & 4n+9 \end{vmatrix}$.

Now $r+s+t$ runs through all residues mod n along any row, column or file (moving two steps at a time so as to keep to cells with coordinates of like parity). Hence the sum of n alternate elements in any row, column or file is either $4n+3$ or $3n-3$ in case (1), or $4n+9$ or $3n-9$ in case (2); and the sum of the other n may be seen to be the complementary number, making the complete sum $7n$ in all cases.

5. The same holds good for the diagonals in case (1), and for three classes of them in case (2). In the latter case $r+s+t$ will along one type of diagonal go through only $n/3$ different residues mod n, all congruent mod 3. But as (2) consists of $n/3$ different sets repeated each three times in succession, this is immaterial.

6. When $n=3$ the above method can be modified to make the cube of order 6 either symmetrical or pandiagonal, and it may be proved that it cannot be both simultaneously.

G. L. WATSON.

2101. *Foci of the sine curve.*

A focus or a directrix of a transcendental curve is entirely a matter of definition. In this note one of the processes for finding the foci and directrices of a conic is formally carried through for the sine curve.

The equation of the tangent at a point (a, b) is

$$y - \sin a = \cos a\,(x-a).$$

The point (p, q) is a focus if

$$x \pm iy = p \pm iq$$

are tangents. Identifying the above,

$$\cos a = \pm i, \quad \pm i(\sin a - a\cos a) = p \pm iq.$$

Thus

$$a = (n+\tfrac{1}{2})\pi \pm i(-1)^n \log(\sqrt{2}+1),$$
$$\sin a = (-1)^n\sqrt{2};$$

and these are respectively the x and y coordinates of the imaginary points of contact of the isotropic tangents. Also

$$p = (n+\tfrac{1}{2})\pi, \quad q = (-1)^n(\sqrt{2}+\log(\sqrt{2}+1))$$

are the coordinates of the corresponding foci.

Thus, it may be said, the foci of the curve

$$y = \sin x$$

are the points

$$[(n+\tfrac{1}{2})\pi, \quad (-1)^n(\sqrt{2}+\log(\sqrt{2}+1))],$$

and the lines

$$y = \pm\sqrt{2}$$

are the directrices.

H. F. SANDHAM.

2102. *The nine-point circles of a quadrangle.*

The following proof of the theorem that

"*The nine-point circles of the four triangles formed by three coplanar points have a common point which also lies on the pedal circle of any one of the points with respect to the triangle formed by the other three*"

may be of interest.

Let the four points be A, B, C, O and let O be taken as the origin in the complex plane and the complex numbers represented by A, B, C be $2a$, $2b$, $2c$. Let z denote the complex variable and primes denote conjugate imaginaries.

Then the nine-point circles of OBC, OCA, OAB are respectively

$$S_1 \equiv bc'(z-b)(z'-c') - b'c(z'-b')(z-c) = 0,$$
$$S_2 \equiv ca'(z-c)(z'-a') - c'a(z'-c')(z-a) = 0,$$
$$S_3 \equiv ab'(z-a)(z'-b') - a'b(z'-a')(z-b) = 0.$$

But, from $S_1 = S_2$ we derive

$$cc'(z-c)(z'-c')S_3 = 0\,;$$

that is, S_3 passes through the second intersection of S_1, S_2.

Consider now the equation :

$$S_1 + S_2 + S_3 \equiv \begin{vmatrix} 1, & 1, & 1 \\ a(z-a), & b(z-b), & c(z-c) \\ a'(z'-a'), & b'(z'-b'), & c'(z'-c') \end{vmatrix} = 0.$$

It is clearly satisfied if we set $z = a+b$, $b+c$, $c+a$, with the corresponding substitutions for z'. Thus the equation of the nine-point circle of ABC is

$$S_1 + S_2 + S_3 = 0$$

and the four circles have a common point.

Turning now, as is sufficient, to the pedal circle of O with respect to the triangle ABC, it is readily verified that the foot of the perpendicular from O to AB is

$$\zeta \quad = (ab' - a'b)/(b' - a'),$$

whence

$$\zeta - a = a'(a-b)/(b'-a'),$$
$$\zeta - b = b'(a-b)/(b'-a').$$

If we substitute these values, with their conjugates, in the determinant

$$aa'S_1 + bb'S_2 + cc'S_3 \equiv \begin{vmatrix} aa', & bb', & cc' \\ a(z-a), & b(z-b). & c(z-c) \\ a'(z'-a'), & b'(z'-b'), & c'(z'-c') \end{vmatrix}$$

we see that the second column is a multiple of the first, and the pedal circle of O is

$$aa'S_1 + bb'S_2 + cc'S_3 = 0$$

which passes through the common point of the nine-point circles.

J. L. BURCHNALL.

2103. *A curious property of the prime number* **503.**

Recently I had occasion to spend some time on an expansion which is a distant relation of Euler's well-known expansion

$$\prod_{m=1}^{\infty}(1-x^m)=1-x^1-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}+\ldots,$$

in which the indices on the right are the so-called "pentagonal numbers", *i.e.* numbers of the form $\frac{1}{2}n(3n\pm1)$. In the course of my work I constructed the set of numbers expressible as $P-Q$ where P assumed all values which were prime numbers not exceeding 1000, and for each P, Q assumed all values which were positive pentagonal numbers less than P. It became important for me to know which prime numbers less than 1000 were not included in the set of numbers which I had constructed. Obviously 997, the greatest prime not exceeding 1000, could not occur; nor could the next greatest prime, namely 991, because the only value available for P was 997, and the difference of these numbers is not a pentagonal number. Likewise, the primes 983, 977, 967, 953 and 947 did not occur; these are all the primes between 947 and 1000 with the exception of $971=983-12$. Below 947 absentees were much less frequent; they consisted of the primes 887, 863, 839, and finally, *longo intervallo*, 503.

It then seemed a natural proceeding to look at the problem from a different point of view, namely, to take any prime p and to determine the smallest positive pentagonal number q such that $p+q$ is prime; for instance, it would appear from the previous statement that, when $p=503$, the value of q exceeds 497.

I have computed the value of q corresponding to every prime p not exceeding 5000, but it seems adequate to give the following table for primes up to 173, together with a few remarks on primes which fall outside the range of the table.

p	q	p	q	p	q	p	q
2	1	31	12	73	40	127	12
3	2	37	22	79	22	131	26
5	2	41	2	83	26	137	2
7	12	43	40	89	12	139	12
11	2	47	12	97	12	149	2
13	40	53	26	101	2	151	12
17	2	59	2	103	70	157	22
19	12	61	12	107	2	163	70
23	126	67	12	109	22	167	12
29	2	71	2	113	26	173	26

It will be observed that the value of q is generally less than p and frequently a great deal less; in fact, the only values of p up to 5000 for which $q>\frac{1}{2}p$ are the following: 3, 7, 13, 19, 23, 37, 43, 73, 103, 359 ($q=210$), 503 ($q=590$).

Of these eleven primes, there are only four, namely, 7, 13, 23 and 503, for which $q>p$. It is not surprising that q should exceed p for several small values of p, but 503 seems out of place with such associates.

An inspection of the values of q for the primes in successive chiliads of integers shows a tendency for the smaller values of q to become scarcer, but with a much smaller tendency for values of q to become larger or for the larger values of q to become more frequent. These assertions are not incon-

sistent because the frequency of the primes in the first five successive chiliads of integers is decreasing.

It does not seem worth while to enumerate the primes for which $q=210$, because there are about a dozen of them ; but the primes for which q exceeds 210 are as follows :

503 ($q=590$), 1259 ($q=222$), 1759 ($q=330$), 1979 ($q=330$), 2039 $q=672$), 2393 ($q=950$), 2819 ($q=222$), 2939 ($q=392$), 3673 ($q=330$), 3691 ($q=442$), 3769 ($q=330$), 4289 ($q=260$), 4643 ($q=260$).

The results which have now been stated show that the value of q corresponding to $p=503$ is relatively quite exceptionally large ; it is not, however, the largest value of q for primes in the range under consideration, in view of the values of q which are associated with 2039 and 2393.

I must express my indebtedness to my colleague Mr. K. L. Wardle for his kindness in checking some of my computations. G. N. WATSON.

2104. *Integration by parts.*

1. The usual statement of the above theorem is :

$$\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx. \qquad \text{.............................(1)}$$

2. In the practical application of the theorem a different statement of the result has certain advantages ;

$$\int PQ\,dx=PQ^*-\int P'Q^*\,dx, \qquad \text{.............................(2)}$$

Q^* denoting $\int Q\,dx$ and P' (as usual) denoting $\dfrac{dP}{dx}$.

Despite its lack of symmetry and the occurrence of the novel symbol *, the form (2) may be found useful for the elementary student of the calculus for the following reasons :

(i) Confusion with the theorem of integration by substitution is less likely.

(ii) The appropriate first step in integrating a product such as $3x^2\cos x$ can be decided on the principle that $P'Q^*$ must be " easier " or " simpler " to integrate than PQ.

(iii) The formula for successive applications of integration by parts is much easier to remember and to apply if this notation is used.

3. The formula referred to in (iii) above becomes

$$\begin{aligned}(PQ)^*&=P\,.\,Q^*-(P'\,.\,Q^*)^*\\ &=P\,.\,Q^*-P'\,.\,Q^{**}+(P'\,.\,Q^{**})^*\\ &=PQ^*-P'Q^{**}+\ldots+(-1)^{n-1}P^{(n-1)'}Q^{n*}+(-1)^n(P^{n'}Q^{n*})^*. \quad \text{...(3)}\end{aligned}$$

The formula (3) is easy to remember since each term (including the term on the L.H.S. and the last term on the R.H.S.) has one more asterisk than it has primes. In practice every term, except the last, is derived from the preceding by differentiating the P factor, integrating the Q factor, and changing sign. Also, since the last term vanishes (apart from an arbitrary constant) if the second last vanishes, it is possible (*e.g.* if P is a polynomial) to write successive terms mechanically till one vanishes.

For example :

$$(x^3e^x)^*=x^3e^x-3x^2e^x+6xe^x-6e^x.$$

H. M. F.

REVIEWS.

Kinematic Relativity. By E. A. Milne. Pp. 238. 25s. 1948. (Oxford)

The theory described in this book was introduced in 1935 with Professor E. A. Milne's *Relativity, Gravitation and World-Structure*. Since then it has been developed by many authors in a large number of papers, the most outstanding of which were again by Milne, published in the *Proceedings of the Royal Society*, and by Milne in collaboration with Dr. G. J. Whitrow in the *Zeitschrift für Astrophysik*. These contributions by Milne have now been rewritten and appear in this one very welcome volume. The book will be of interest to those who are new to the subject, as well as to those who have followed Milne's work in the past fifteen years. Although a sequel to *World-Structure*, it is not properly dependent upon the earlier book; Milne does not amend anything he said then, but in *Kinematic Relativity* he is concerned with strengthening the basis of the theory as well as with describing its development in directions unforeseen when *World-Structure* was written. To those who followed the work as it came out this book is of interest, because it is not a mere repetition of what has already been published. In presenting the theory as a whole, Milne has taken the opportunity of polishing earlier methods, and at times of constructing new arguments in the light of later experience.

Kinematic Relativity is divided into four parts: Kinematics, Dynamics, Gravitation, and Electrodynamics. Part I starts with a chapter describing in general terms the ideas underlying the theory, and then gives a full account of the investigations by Milne and Whitrow on time-keeping and clock regraduations. This leads to the derivation of the Lorentz equations relating measurements made by different observers of an " equivalence ", at first in one dimension and then in three. When these results are applied to the system of fundamental particle-observers, supposed situated at the centres of the galaxies, there emerges a picture of the *substratum*, or model of the whole smoothed-out universe. This is the material system which, in agreement with Mach's Principle, is always present as a background in the later gravitational theory.

In Part II is constructed a formal dynamics closely analogous to the classical theory, and it is shown that a potential theory can be constructed so that it has the required relativistic invariance. The significant results are the equations of motion in a conservative field, with a given scalar potential; and these are applied in Part III to the study of certain statistical systems of particles. There emerges a gravitational theory with an approximate inverse square law, the classical analogy being made still closer when the time-scale is transformed from t- to τ-measure; as a consequence the latter is identified with the gravitational time-scale. Part III concludes with an account of Milne's recent work on the structure of spiral nebulae; the similarity between the theoretical shapes of spiral arms and those actually observed is most striking, particularly in the case of the barred spirals.

In Part IV is formulated an electromagnetic theory based upon the dynamics of Part II. Modified forms of Maxwell's equations are derived, and the interaction of two point-charges is found to be strictly inverse square. An interesting result, reminiscent of the classical " radius " of the electron, is that every point-singularity is found to have an apparent " radius ". In the final chapter Professor Milne gives a welcome survey of the achievements of the theory, unobscured by the details of the previous investigations. The book finishes with a comprehensive bibliography and an index.

It is difficult in the space of a short review to give an adequate account of all that the book achieves, but there are three outstanding contributions to knowledge that may be described briefly. Firstly, there is the study of time-

scales and clock regraduations leading eventually to the two fundamental scales of t and τ connected by an exponential relation. The suggestion that natural clocks do not all keep the same kind of time is perhaps the most striking in the whole theory ; the search for a conclusive test may well become an outstanding problem of experimental physics. One consequence of the hypothesis that there are two natural time-scales is a duality running through all descriptions of physical systems because of the way these descriptions depend upon the choice of time-scale. For example, with the t-scale the universe is expanding, but with the τ-scale it is static, an unexpected answer to the old question as to whether the universe is expanding or not.

The second contribution is in showing that the cosmological problem, of constructing a model for the whole universe, can be approached by simple and direct methods without all the complicated geometry and field equations of other relativity theories. This, of course, was the main purpose of *World-Structure*, but it can now be seen much more clearly. An attractive feature is that the final structure, the substratum, is a single model, with no constant remaining to be determined empirically. This should make it possible to test the theory by means of observations when these are rather better than at present. The same can hardly be said of General Relativity, for example, because of the appearance of an unknown function in the general solution.

The third contribution is in showing that it is possible to redefine objects such as momentum, force, energy, potential, etc., so that they are analogous to the classical objects and yet are relativistically invariant. This is a direct consequence of the fact that observations are supposed made only by the fundamental observers, so that physical equations must be invariant under a group of transformations much more restricted here than in other relativity theories. With the introduction of the τ-scale of time the classical analogy is still closer ; space-time separates into space and time, the Euclidean space of Newtonian mechanics now being replaced by a " public " (*i.e.* invariant) 3-space of constant negative curvature, and time is again a parameter. One result of this return to classical form is that we may again hope to use the large body of potential theory in the study of gravitational and electromagnetic problems.

This book will doubtless be a stimulus to further developmental research in Kinematical Relativity. It is also to be hoped that some of the earlier work will receive further attention, for although Milne sets out to be strictly deductive, he is at times intuitive, taking one path when there are others also consistent with the fundamental principles of the theory. Thus some of the results do not at present satisfy the test of inevitability ; some new idea, a new principle perhaps, may be necessary to make the argument rigorous. For example, the important formulae $M=m\xi^{\frac{1}{2}}$ (equivalent to the formula $m(1-v^2/c^2)^{-\frac{1}{2}}$ of Special Relativity) and $\Omega=Mc^2$ for relative mass and energy appear in Sections 76 and 87, and we would like to see these established as consequences of the theory. But we find that the first is adopted as a *definition* of relative mass, and the second is really assumed at the beginning of a circular argument. It looks as if the theory needs a deeper analysis of the definition and measurement of inertial mass. Another example is the definition of potential. This is not inevitable because of the way it is based upon a purely formal definition of force. Other equally valid definitions of potential can be constructed, and some other definition may in fact be desirable, because at present the equations of motion in a conservative field do not satisfy a variational equation.

These comments, while suggesting that the work is not yet complete, should not be allowed to detract from the very great value of *Kinematical Relativity.*

In presenting the essential ideas of his theory, Professor Milne has shown us one way in which they may be developed mathematically, and his book will undoubtedly be a powerful stimulus to further research in this fascinating field.

A. G. W.

Cardinal Algebras. By A. Tarski (with an Appendix on Cardinal Products of Isomorphism Types by B. Jónsson and A. Tarski). Pp. xii, 326. 50s. 1949. (Geoffrey Cumberlege, Oxford University Press)

This work is unreadable in the sense that a well-trained mathematician working in a field different from that of the author might fairly regard the effort of reading it as out of all proportion to the profit to be derived from mastering its contents, and a technical expert will probably prefer it in small doses. This, of course, is not intended, and cannot fairly be interpreted, as a complaint against the author. The work is highly technical, and in order to confine it within a reasonable compass, while satisfying the highest canons of rigour and accuracy, the author has no choice but to fire a terrifying barrage of Theorems, Corollaries and proofs.

A Cardinal Algebra is, like other algebras, a system with two rules of composition. One of these is a binary operation of addition. The other is an operation which associates with every enumerable sequence of elements of the algebra (repetitions being allowed) another element which is the sum of the sequence. The difference between this and other algebras is, firstly, that one of the operations is not finitary (in the sense of requiring only a finite number of elements to enable it to work); and, secondly, that, since the existence of zero is postulated, the finitary operation is, in a sense, implied in the other one.

These algebras are introduced in order to study that part of the theory of cardinal numbers which can be derived from a few basic ideas, here taken as postulates in the algebra, without the use of the axiom of choice, *i.e.* using only what may be roughly described as non-theological methods. Naturally an important result of this abstractification of a well-established theory has been to extend its range of application. Even so, readers with a good background of abstract algebra (the subject to which this work formally bears allegiance) may find themselves in a territory where very few of the landmarks seem familiar. The book will concern chiefly those interested in logic and abstract set-theory, and in these fields it may be expected eventually to have a profound influence.

D. B. S.

Pfaff's Problem and its Generalizations. By J. A. Schouten and W. v. d. Kulk. Pp. xii, 542. 50s. 1949. (Geoffrey Cumberlege, Oxford University Press)

The problem treated in this book has been the subject of a large number of papers and books for the last hundred years. Such well-known mathematicians as Gauss, Jacobi, Grassmann, Frobenius, Lie, von Weber, Goursat and Cartan have made contributions to its study. This book contains a detailed survey of theories on Pfaff's problem and its known generalizations. The two authors themselves made some further generalizations during the last war, and their results are included in two of the later chapters. This book differs in presentation from any of the existing books on the subject in that the kernel-index method of tensor calculus is used throughout. The notation, terminology, and general approach is based upon Schouten's fundamental contributions to tensor calculus and differential geometry.

The first eighty pages are devoted for the most part to the study of those parts of the tensor calculus which have already appeared in various works by Schouten, and which are essential for the understanding of the remainder of

the book. This part is carefully selected, and the reader unacquainted with tensors can master the essentials without reference to any other book.

Starting from the arithmetic manifold of points in n dimensions (ξ^κ) $(\kappa = 1, 2, \dots, n)$, this manifold of points together with the pseudo-group of all analytic invertible transformations between allowable coordinate systems, constitute a geometric manifold X_n. Considering the transformation of the differentials $d\xi^\kappa$ at any point,

$$d\xi^{\kappa'} = (\partial_\kappa \xi^{\kappa'})_0 d\xi^\kappa = A_\kappa^{\kappa'} \,.\, d\xi^\kappa,$$

we notice that the pseudo-group of all allowable coordinate transformations induces an affine group at every point. Hence to every point of X_n there is attached a local affine space E_n. The geometry of an E_n is treated in the first chapter. If quantities such as scalars, vectors, p-vectors, are defined in all local E_n's of the points of a certain region of a point, they constitute a ***field***.

An expression such as $w_\lambda d\xi^\lambda$ where w_λ is a covariant vector field is called a Pfaffian, and the equation $w_\lambda d\xi^\lambda = 0$ a Pfaffian equation. Writing

$$W_{\mu\lambda} = \partial_\mu w_\lambda - \partial_\lambda w_\mu,$$

if the condition

$$W_{\mu\lambda} w_\nu + W_{\lambda\nu} w_\mu + W_{\nu\mu} w_\lambda = 0$$

happens to be satisfied, the Pfaffian equation is said to be completely integrable and there are ∞^1 integral X_{n-1}'s such that their tangents are in the $(n-1)$-direction at every point. In the terminology of the authors a Pfaffian equation represents an E_{n-1}-*field* in an X_n, and if the equation is completely integrable, the E_{n-1}-field is X_{n-1}-*forming*. If the equation is not completely integrable, the problem becomes that of finding X_m's $(m < n-1)$ such that the tangent E_m at every point lies in the E_{n-1} of the field at the point.

We now introduce two definitions: (i) an X_m $(m \leqslant p)$ is said to be *enveloped by* an E_p-field or to be an *integral* X_m of the E_p-field if the tangent E_m is contained at every point in the local E_p of the field; (ii) an X_m $(m \geqslant p)$ is said to *envelop* an E_p-field if its tangent E_m contains, at every point, the local E_p of the field.

The so-called *simple* Pfaff's problem is that of finding, for $p = n-1$, all the *enveloped* X_m's. The first generalisation of the problem is that of considering a system of $(n-p)$ linearly independent Pfaffian equations which represent an E_p-field in the X_n. In this case there are two problems.

The *inner* problem requires the determination of all the X_m's $(m < p)$ which are enveloped by E_p's of the field for the *maximum* value ν of m. The *outer* problem requires the determination, for the *minimum* value of m, of all the *enveloping* X_m's $(m > p)$. Only when ν for the inner problem is equal to p do the two problems coincide. In that case the inner problem is solved if we have a solution of the outer problem. When $\nu < p$ the relation between the two problems is more complicated.

The authors have given a formulation of both the inner and the outer problem which is suitable for generalisation. The generalisation is stated in terms of geometric objects, of which particular examples are E_n's, scalars, vectors, p-vectors, etc. The treatment given to the well-known theories of Cartan systems and Goursat systems leads naturally to the generalisations carried out by the authors themselves. An important part is played in the treatment by supernumerary coordinates, well-known examples of which are the $(n+1)$ projective coordinates of a point in n-dimensional projective space, and the pentaspherical coordinates of Darboux in ordinary space. Their advantage is that only by their use can the invariance of certain equations be preserved.

It would be impossible in a short review to give even the beginning of an

indication of the large number of concepts, operations and results given. The authors point out that in order to give the solution of the problem in the general form which they have given to it, they would need to develop the whole of the theory of differential equations and much more. There is therefore still a wide range of problems in this branch of mathematics. It is certain that any worker in this field will find this book an indispensable companion for many years to come. E. T. DAVIES.

Introduction to Algebraic Geometry. By J. G. SEMPLE and L. ROTH. Pp. xv, 446. 30s. 1949. (Clarendon Press, Oxford)

" The main object of this book is to provide a reasonably concise introduction to algebraic geometry, requiring no more background than the usual honours degree courses in projective geometry and algebra. We have tried especially to satisfy two needs : (i) to give the reader an adequate idea of the developments in the past hundred years, and (ii) to provide him with every opportunity, in the form of examples, for acquiring self-reliance and technical ability." This extract from the authors' preface describes perfectly the scope and aim of the book, which might appropriately bear the sub-title, " What every young geometer ought to know." After a short introductory chapter there is a concise account of the projective theory of higher plane curves, and a study of the plane quadratic transformation (with applications to the resolution of singularities). Next comes a rather long chapter on rational correspondences (the problems arising from multiple coincidences being discussed with care), with applications to the relations between the projective characters of curves in a plane and in space, and the determination of the characters of the residual curve of intersection of two surfaces with a common base-curve. This is followed by an account of the theorems of Nöther and Serret, and two chapters centring round the notion of the projective model of a linear system of plane curves. Here the reader is introduced, by way of their plane representations, to most of the well-known rational surfaces of low order. This leads quite naturally to an account of the representation of rational threefolds by linear systems of surfaces, and to some typical Cremona transformations in three and four dimensions. The next chapter deals with various enumerative properties of curves and surfaces, and introduces the numerical invariants associated with a surface. A chapter on line-geometry and Grassmannians follows ; this is succeeded by an account of degeneration methods and the calculus of conditions due to Schubert. Finally, there are two chapters on birational geometry, the first being a concise but fairly thorough account of the theory of linear series on a curve, leading to the Riemann-Roch theorem and the study of valency correspondences, while the last chapter in the book gives a brief sketch of the vastly more difficult theory of algebraic surfaces. There is a detailed table of contents, but unfortunately no index.

As befits a work of this nature, whose prime object is to provide information about geometrical facts, the authors have adopted the traditional and somewhat naïve attitude to the foundations of the subject. This is an entirely reasonable course to take, especially when, as frequently in the present work, some warning is given when the authors propose to tread on delicate ground. The serious student of the rigorous algebraic foundations of the subject will gain rather than lose by a broad acquaintance with the main results which will ultimately rest on these foundations. In particular, the authors' dissection of the principles involved in the application of degeneration methods, though it quotes general theorems without proof, is a very skilful piece of work, and leaves no doubts as to the nature of the points at issue.

There are points at which the naïveté seems to be carried to somewhat

excessive lengths, and the reader who has mastered the definition of an irreducible algebraic manifold as the birational transform of a primal may well wonder how to justify the assertion that the irreducible components of the intersection of a set of primals are in fact irreducible algebraic manifolds. But the authors' occasional lapses of this nature are more than counterbalanced by the excellence of the general exposition. The summary of the contents given earlier in this review will indicate in some measure the range of this work, but cannot give any hint of its power and inspiration. For inspired it certainly is. The spirit of H. F. Baker underlies almost every page, and the whole volume is a striking monument to his work for English geometry. To the reviewer, at least, this work has evocative qualities not surpassed by the writings of Baker himself: to understand, and to perceive the unity in diversity in its various chapters, is to come near to comprehending the impulses which activated the school of geometers whom Baker taught. The authors have done full justice to the elegant nature of their subject, both in their treatment of general theory and in the careful selection of their numerous illustrative examples. It is the reviewer's hope that this book will be read by many of our younger mathematicians, and his belief that it will turn many of its young readers into serious geometers. It would certainly be difficult to conceive a more stimulating introduction to algebraic geometry, or one which reveals so clearly the richness and beauty of the fields to which it opens the way.

At a time when interest in algebraic geometry seems to be increasingly directed to an examination of its logical basis, a work such as this may be regarded by some as an anachronism. This view is surely mistaken; a mathematical theory consisting of foundations with no superstructure is sterile. That in the growth of the subject during the present century the superstructure has been extensively developed while the foundations were left in a somewhat unstable condition may be admitted, but the very extent of this growth bears witness to the fascination which the existing theory presents, in the remarkable interrelations of its various aspects, and in the tantalising problems which it presents, the attempts to solve which seem inevitably to lead to extensions of its scope. In such a field there is always room for a book which will introduce the reader to the main features of the subject in such a way as to stimulate his interest. In this respect the authors have succeeded admirably, and the book is likely to become a classic. It will certainly have been read with delight by most geometers by the time this notice appears in print.

The Clarendon Press have, as usual, produced a work of pleasing appearance. There seem to be few misprints that matter; but the reviewer failed to understand a reference to a paper of his own on p. 268, and is curious as to the contents of the hitherto unknown *fifth* volume of the treatise by Enriques and Chisini mentioned in the bibliography. J. A. TODD.

Le superficie algebraiche. By F. ENRIQUES. Pp. xv, 464. Lire 3000. 1949. (Zanichelli, Bologna)

Federigo Enriques, at the time of his recent death, had stood for over half a century in the first rank of algebraic geometers. The algebro-geometric development of the theory of algebraic surfaces, which is the subject of the volume under review, is the result of the work of many mathematicians, but the three names of Castelnuovo, Enriques and Severi stand out as its principal architects, and almost all the principal theorems of the subject are due to one or other of these three great geometers.

The manuscript of this posthumous work appears to have been virtually complete at the time of the author's death; it has been seen through the press by his pupils Pompilj and Franchetta, and a moving preface has been

contributed by Castelnuovo. It is essentially a revised edition of an earlier work, edited by Campadelli, which appeared in two parts; the first being the lithographed *Lezioni sulla teoria della superficie algebriche* published in 1932, and the second being published in Rome two years later. The former, at any rate, of these two volumes is well known to English geometers, though it seems to have escaped notice in the *Mathematical Gazette*. The revised version at present under review covers much the same ground as the two earlier volumes, but is rather more detailed. The eleven chapters which follow a short introduction deal with linear systems of curves, invariant and covariant systems, adjoint surfaces, the arithmetic genus and the Riemann-Roch theorem for surfaces, numerical invariants and multiple planes, regular surfaces and conditions of rationality, surfaces with linear genus one, regular canonical and pluricanonical surfaces, irregular surfaces and continuous non-linear systems of curves, surfaces of geometric genus zero, and the general classification of surfaces.

The book thus covers most of the principal aspects of the theory, with some emphasis on the difficult problem of classifying surfaces by means of their invariants, a subject which Enriques made peculiarly his own. The most obvious omissions are any mention of the theory of the base, which presumably falls rather outside the author's scope since it rests in part on transcendental considerations, and Severi's theory of systems of equivalence of sets of points, for which the reader is referred to Severi's own book on the subject. The standpoint throughout is that of the classical Italian school; there is no proof of the fundamental result that an algebraic surface can be birationally transformed into one without singularities (though adequate references are given). The author has, however, taken some pains to draw attention to the numerous delicate points which arise in various parts of the theory; in particular Chapter IX contains a somewhat detailed account of the many unsuccessful attempts to prove, by algebro-geometric methods, the theorem on the completeness of the characteristic series of a complete continuous system of curves in its most general form. This now notorious problem, which has so far defied the efforts of geometers of the calibre of Enriques and Severi, presents perhaps the most arresting challenge in the whole algebro-geometric theory; it is a merit of Enriques' account that the essential difficulties are clearly set out.

This treatise, embodying the fruits of a lifetime's labour by its author, gives a clear account of the present state of the theory of algebraic surfaces as developed by the methods of the Italian school, and will be a standard work of reference in the field for a considerable time. J. A. TODD.

La Géométrie Intégrale du Contour Gauche. By A. BLOCH and G. GUILLAUMIN. Pp. vi, 141. 1500 fr. 1949. (Gauthier-Villars)

The language of mathematics appears to have split into a number of dialects in such a way that it is usually difficult for students in different fields to hold converse with one another. It is therefore especially refreshing to find a work of this kind speaking a language which all mathematicians can understand and which most will be tempted to claim as their own. This charming little book will resist all the attempts of examiners or others to classify its contents even under the broadest possible headings of "pure" and "applied". The subject-matter, of closed curves in ordinary space, is simple and concrete enough to satisfy the most physically-minded that the subject is significant, and subtle enough to satisfy the most abstractly-minded pure mathematician that it is worthy of study, even without the author's generalisations (in the final chapter and appendices) to non-euclidean geometry and to n-dimensional space.

One of the book's most delightful features is the comfortable feeling of familiarity which it gives the reader, while taking him into realms which are new to him. The methods used all remind us of things we have seen before, and it is most instructive to see them set to work in different fields. The student of mechanics or electricity may at first find himself more at home than the geometer who, however, cannot fail to be delighted at the effective use of statical ideas. These arise very naturally in the attempt to associate descriptive numbers with a closed curve. The simplest such set of numbers is the vector area. This, however, tells us very little about the curve, and further information can be derived from the integrals of the first and second order along the curve, which give the moments, and moments and products of inertia, of the curve (regarded as a uniform wire) relative to the axes of reference. Another approach is to consider the effect of applying a uniform pressure to any surface bounded by the curve. The system of forces thus obtained (which by a well-known theorem in hydrostatics is independent of the particular surface chosen) gives us a wrench (in the sense of Lamb's *Higher Mechanics*). This naturally is tied up with properties of the curve, the vector area being in fact the resultant force of the system. Equally we can consider electrical ideas such as the effect on the curve, regarded as a wire carrying a uniform current, of a uniform electro-magnetic field. All these ideas and many others are employed in the discussion, and the more sophisticated geometer will be delighted by the production in later chapters of an algebraic line congruence associated with the contour, which is dual to the congruence of chords of a twisted cubic.

In short, this book contains something for everyone with an interest in mathematics and some knowledge of vector analysis. If the line congruence is too highbrow, there is plenty of good stuff before that is reached. Of course, it is not all armchair reading. But the authors provide good value for whatever effort you may care to put into reading the book. D. B. S.

Introduction to Applied Mathematics. By F. D. MURNAGHAN. Pp. ix, 389. 30s. 1948. (John Wiley, New York; Chapman and Hall)

This book is the first of a series to be devoted to mathematical theories underlying physical and biological sciences and with advanced mathematical techniques needed for solving problems of these sciences. Other volumes in preparation include such topics as Numerical Methods, Thermodynamics, Complex Variable Theory and Mathematics of Relativity.

The volume under review, contrary to the impression created by the title, deals almost entirely with the methods of Applied Mathematics as distinct from the principles and developments of the subject. There are, however, a few examples taken mainly from electrostatics and dynamics to illustrate the methods evolved. It appears that the contents of the work comprised a course of lectures given by the author to graduates in the faculty of science at the Johns Hopkins University. The subject-matter ranges from vectors and matrix theory to the Calculus of Variations and the Operational Calculus with a definite bias towards the use of vectors and matrices wherever possible.

In the first chapter the reader is introduced, from first principles, to elementary vector theory, including the elements of vector field theory in two and three dimensions. In the same chapter we find a little of the elements of matrix theory introduced to define an n-dimensional complex vector. The rest of the chapter is devoted to developing a few of their properties. Chapters 2 and 3 deal entirely with vector functions, including such conceptions as Hermitian and skew symmetric functions culminating in orthonormal sets, formal Fourier analysis, and linear integral operators. Reading on, we find that Chapter 4 deals with orthogonal curvilinear coordinates along with the

appropriate expansions for the vector field functions, whilst Chapter 5 is devoted to Laplace's equation and its solutions with examples taken from electrostatic theory as illustrations. In the same chapter we find an explanation of the method of Inversion, and also an interesting account of ellipsoidal coordinates. (It should be noticed on page 160 that the expression

$$\lambda + \mu + \nu = x^2 + y^2 + z^2 = r^2$$

should read
$$\lambda + \mu + \nu = x^2 + y^2 + z^2 - a^2 - b^2 - c^2.)$$

Following the account of Laplace's equation in Chapter 5, in Chapter 6 the author gives an ordered account of spherical harmonics and Bessel functions, together with a discussion on the zeros of the latter. Boundary value problems together with the allied Green functions are discussed in Chapter 7, whilst in Chapter 8 the author gives an account of some integral equations culminating in an interesting and very welcome exposition of Rayleigh's principle. The last two chapters conclude the volume with the fundamentals of the calculus of variations and operational calculus respectively.

It is obvious from the brief account given of the contents that Prof. Murnaghan has covered a wide ground in his book, and I think he has certainly reached his objective in satisfying some of the initial needs of a young physicist or applied mathematician from the point of view of providing an organised account of the essential mathematics. Whilst the book, in the main, is written in a lucid style, there are certain sections where I think the beginner would find it hard going. For example, the theory of n-dimensional complex vectors could easily have been included in a separate chapter along with a little more matrix theory as a preliminary and adequate introduction. The act of the author in cramming a tremendous amount of vector theory into the rather short space of the first chapter is a serious drawback for the complete understanding of Chapters 2, 3 and 4. Another, and, to me, a surprising feature of the book was the absence of any worthwhile complex variable theory, so necessary to the present-day mathematician. I think, too, that the last chapter would have benefited greatly from the inclusion of the Laplace-transform inversion theorem.

I should add, in conclusion, that there are hundreds of exercises given, both to test the reader's understanding of the theory and to develop further certain ideas. There is also an adequate index containing references to the text. The printing and general appearance of the volume are excellent. J. W.

Cours de Mécanique. Tome I. Par Henri Beghin. Edition provisoire polycopiée. Pp. 588. 1800 fr. 1948. (Gauthier-Villars)

This work, the author tells us in his preface, contains the substance of lectures delivered by him at the École Polytechnique. The present volume covers roughly the first year's work. The topics considered are indeed much the same as we find in, say, Painlevé's *Cours de Mécanique*, but how different the treatment and how singularly novel. One may well be surprised, after a glance through the book, to find the discussion of the motion of a particle deferred till after the motion of a rigid body in three dimensions. But, the author says in his introduction, his aim has been " to provide the reader in the simplest form the means to *foresee* the phenomenon of mechanics in a way designed to engender the minimum intellectual effort ". The result is an interesting departure from current treatment, and one which has apparently met with some success.

Instead of the usual synthesis beginning with the motion of a particle and gradually leading up to the motion of a rigid body in three dimensions, the author develops first the principles of mechanics in the most general manner and then proceeds to illustrate each principle by means of examples of many

types drawn from statics, dynamics and hydromechanics. Thus, in Chapter VII, under the subheading " Exemples simples sur le théorème du centre de gravité et sur le théorème de la somme des quantités de mouvement ", we find the following problems discussed, *inter alia* : Projectiles. Recoil of a gun. Ascent of a cork in a vessel full of water. Bird in a cage. Impact of a liquid jet on a fixed surface. Principle of the rocket.

This first volume appears to fall into two parts, though the author makes no such division. In the first part the general principles of mechanics are developed, beginning with a preliminary chapter on kinematics. This contains an extensive account of the composition of motions. We then have chapters on centres of mass and quadratic moments of distributions of masses, and on the laws of motion, including a brief historical sketch. One of the most interesting and illuminating chapters deals with Galilean Systems—a topic which is either lightly passed over or else entirely ignored in most textbooks on Mechanics. After a discussion on the properties of the actions of surfaces in contact, we have two entire chapters of examples illustrating the various principles of Mechanics. The first part ends with a discussion of Work and Power and the motion of a particle. The long chapter on this topic is particularly good, and one interesting example shows that in certain cases the equations of motion of a particle (in three dimensions) may possess several solutions.

What may be called the second part of the book opens with a chapter on Virtual Work. Here also many examples are worked out, often with an alternative solution making use of a " velocity diagram ". There follows the analytical part of dynamics, including a chapter on Appell's equations, and here standard treatments are used. Throughout the work the author's outlook is that of the physicist rather than the mathematician.

In his foreword General Brisac points to three reasons for the success of the work : simplicity, freedom from dogma, and a sense of the concrete. Certainly his thesis is well supported by the many excellent pages that follow. And whatever may be one's reaction after reading this book, few will deny that M. Beghin has written a book on Mechanics which can truly be said to be " new ".

There are numerous interesting examples at the end of most chapters, and many appear to be novel. V. C. A. FERRARO.

Principes de la mécanique classique. Par J. L. DESTOUCHES. Pp. 137. 1948. (Centre National de la Recherche Scientifique, Paris)

This book is the first of a series on mathematical physics which are to be published by the " Centre d'études mathématiques en vue des applications " in Paris. In an editorial preface, J. Cabannes indicates that the purpose of the " Centre " is the essentially practical one of enabling physicists to solve mathematical problems, arising in the course of their work, with the aid of the most appropriate current theoretical concepts and techniques. In pursuit of this purpose, the subsequent related contributions to the series, which apparently will deal with wave mechanics, statistical mechanics and relativistic mechanics, are evidently intended to be of a more practical character than the present one. However, since Newtonian mechanics is the parent of all these other systems, the authorities of the " Centre ", and M. Destouches in particular, consider that the principles of Newtonian mechanics ought to be set out with a degree of rigour and completeness that is not contemplated for the presentations of the other systems.

The postulates have to be given in full in order that the restrictions of Newtonian theory may be made clear, and also in order that it may later be shown how they are relaxed or replaced in the newer theories. The basic

theorems have to be presented because they have such a profound influence upon the formation and development of these theories. This latter consideration has obviously weighed a great deal with the author, though he may not state it in so many words. One believes it to be of great importance. For one is under the impression that " general " or " analytical " dynamics is much less studied at the present time than it was in the early days of the relativity and quantum theories. So the younger students of these theories may not be in a position fully to appreciate some of their formulation without recourse to some such specially designed presentation of the classical background as M. Destouches has sought to provide. Whether he has made the most profitable selection of the parts of classical theory for this purpose is another matter.

The first of the three chapters of the book gives a careful and orderly presentation of the concepts and postulates of classical mechanics. Or rather, it gives one possible set of postulates, for there is, of course, considerable arbitrariness as to what propositions are to be regarded as postulates: for instance, one may postulate the existence of *inertial mass* or, as the author does, deduce it from the postulated " principle of action and reaction " and certain allied " principles ". The set of postulates chosen by M. Destouches is generally speaking along the lines first laid down by Mach: a shorter and less formal recent account of essentially the same system occurs in E. A. Milne's *Vectorial Mechanics* (Chapter XI).

While it is very useful to have this accessible treatment of the postulates themselves, one wonders if it provides all that is needed in preparation for other theories. One would have thought that an analysis of the extent to which the subject can be regarded as derived from experience would be rather more important. One has in mind the sort of development sketched in H. Jeffrey's *Scientific Inference* (Chapter VIII).

The other two chapters are on fundamental theorems of classical mechanics and on " first integrals ". They contain a useful digression on the problem of how a " fundamental " or " unaccelerated " frame of reference is to be determined in the actual universe. A good deal of attention is given to the representation of a dynamical system in configuration space. These particular topics receive special treatment presumably on account of their importance in relation to the foundation of relativistic and statistical mechanics, respectively. However, the author has not found space for any of the theorems of what is usually known as " analytical dynamics " (associated with the names of Hamilton, Jacobi, Poisson and others). In view of their analogues in quantum mechanics, one is surprised at the omission.

At the end of the book a number of simple standard examples of particle dynamics are worked out so as to illustrate as much as possible of the foregoing theory.

As signified on the title page, the final form of the text has been discussed with many of the author's colleagues, whose names are given. These are evidently the persons who will be responsible for the other related contributions to this series of texts. M. Cabannes emphasises in his preface that the present contribution is not to be judged in isolation from these others, for which it prepares the way. Therefore the merits of the work under review will have to be assessed in retrospect after the rest of the series has been published.

W. H. McCrea.

Dialectica. 7/8. 1949. (Griffon, Neuchâtel)

This issue of the quarterly review *Dialectica* is devoted to contributions on " The Concept of Complementarity ". The concept was first explicitly formulated by Neils Bohr in order to describe the novel type of relationship

between the results of observing atomic objects under different experimental conditions, which is an essential consequence of the recognition of quantum processes. It follows from the impossibility of rendering arbitrarily small the interaction between an atomic object and the measuring instruments employed to determine its behaviour, owing to the finiteness of the quantum of action.

The first article is by Bohr himself, and gives a careful statement of the necessity for the concept and a demonstration of how the quantum-mechanical formalism is exactly adapted to express the complementary mode of description.

In the second article, Einstein states why he regards the methods of quantum mechanics as unsatisfactory in their current form, and as destined to be superseded by some more profound or comprehensive system.

The remaining contributions by L. de Broglie, W. Heisenberg, H. Reichenbach, J. L. Destouches, P. Destouches-Février and F. Gonseth discuss so many physical and metaphysical aspects of the subject as to touch upon almost every fundamental problem of natural knowledge. The opinions of so many distinguished thinkers will be valued by all who are interested in the wider implications of the concepts of quantum theory.

It is unusual to review journals in the *Gazette*, but the present case is exceptional since it concerns a set of essays rather than " communications " to a periodical. One refrains from a fuller review, not because it would be inappropriate, but merely because it would be superfluous. For the essays are issued under the editorship of W. Pauli, and his brilliant " Editorial " is itself an ample review of the whole set.

One would wish further only to draw particular attention to Pauli's concluding remark " that the present quantum theory, which is insufficient to explain the atomistic nature of electricity and to predict the mass values of the ' elementary ' particles in nature, can only have a limited range of application. We are here only at the very beginning of a new development of physics. . . ." Thus, so far as metaphysics is concerned, although the present quantum theory has made a permanent contribution by revealing the inadequacy of certain earlier concepts, it almost certainly ought not to be relied upon to provide the foundation of a new metaphysical structure.

W. H. McCrea.

Éléments de la Théorie des Ensembles. By É. Borel. Pp. 319. 720 fr. 1949. Bibliothèque d'Éducation par la Science. (Éditions Albin Michel, Paris)

Any new publication by the famous doyen of the French mathematicians on the theory of point sets, the development of which to its present state of mathematical importance owes so much to his own fundamental contributions, is bound to be of the highest interest. The present little book is intended as an introduction into the elements of this theory. It covers, of course, the more usual features of the subject, restricting itself, wisely, mainly to linear sets. Enumerable and non-enumerable sets, the ternary Cantor set, the Peano curve are discussed, and the book then steps up to the theories of measure according to Jordan, Borel and Lebesgue. However, as one would expect with Borel, the main interest of the book lies in what it contains besides (partly in the additional notes). There is, firstly, the aspect of the theory of measure as a theory of probability, *i.e.* of the probability that a point belongs to a given set. To this part belongs Borel's theory of " enumerable probability ". Next, Borel is the senior leader of the French " constructive " mathematicians, other famous representatives of which are Baire and Denjoy. He prefers, in a rather ostentatious way, definitions and arguments, which can be " realised " by an at most enumerable sequence of con-

structive steps, to such as are merely theoretical and based on abstract properties of the continuum. In consequence we find a thorough discussion of linear Borel sets (usually written as sets of decimal fractions) which are constructable in this sense. The theory of " rarefication " of such Borel sets of measure zero is another interesting item not usually touched in textbooks. The book ends with a critical discussion of the " axiom of choice ", which is, as one would expect, severely dealt with.

The book is written throughout in the usual temperamental Borelian style, which has lost nothing of its freshness. It is, I think, not a book easy to read for a beginner. For part of its contents is not elementary enough, and the wide ground covered has forced the author to a sketchy representation. Nor is a beginner in a position to view critically the polemic attitude of the book. For the more experienced reader it is a delightful little book. Unfortunately, the text is impaired by an unusually large amount of misprints, some of which make the reading rather difficult.

W. W. Rogosinski.

NATIONAL BUREAU OF STANDARDS MATHEMATICS LABORATORIES

National Bureau of Standards, Applied Mathematics Series

1. **Tables of the Bessel Functions** $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leqslant x \leqslant 1$. Pp. x + 62. 1948. 35 cents.

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5. **Table of Sines and Cosine to Fifteen Decimal Places at Hundredths of a Degree.** Pp. viii + 96. 1949. 40 cents.

All prepared by, or under the auspices of, the Computation Laboratory of the National Applied Mathematics Laboratories, under the supervision of Dr. A. N. Lowan, Chief of the Computation Laboratory. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C. Paper covers. $10\frac{1}{4}'' \times 7\frac{3}{4}''$ (26 cm. × 20 cm.).

The Applied Mathematical Series is intended to serve as a vehicle for the publication of shorter mathematical tables, manuals and studies by the mathematics laboratories of the U.S. National Bureau of Standards. The mathematical tables in the series represent a continuation of those formerly published in various journals, particularly in the *Journal of Mathematics and Physics*, issued by the M.I.T.

All the booklets under review were prepared by the Computation Laboratory, formerly the Mathematical Tables Project, but now absorbed into the comparatively recently formed National Applied Mathematics Laboratories. This review thus continues a series describing tables prepared by this organisation of many names (see *Mathematical Gazette*, **29**, 1945, pp. 29–33, 86–87, **30**, 1946, 49–52, 239–242, **31**, 1947, 181–184, **33**, 1949, 70–72).

(1) This gives 8 or 9 figure values of $Y_0(x)$ and $Y_1(x)$ for

$$x=0(0{\cdot}0001)0{\cdot}05(0{\cdot}001)1,$$

with Δ and Δ^2 for $x>0{\cdot}005$. Values of $K_0(x)$ and $K_1(x)$ are to 7 figures for $x=0(0{\cdot}0001)0{\cdot}033(0{\cdot}001)1$, also with Δ and Δ^2. For small values of x, where second difference interpolation is inadequate, auxiliary functions are given: $C_0(x)$, $D_0(x)$, $C_1(x)$, $D_1(x)$ to 8 decimals, with Δ, for $x=0(0{\cdot}0001)0{\cdot}005$, where

$$Y_0(x)=C_0(x)+D_0(x)\log_{10}x,$$

$$Y_1(x)=\frac{C_1(x)}{x}+D_1(x)\log_{10}(x);$$

also $E_0(x)$, $F_0(x)$, $E_1(x)$, $F_1(x)$ to 7 decimals with Δ for $x=0(0{\cdot}001)0{\cdot}03$, where

$$K_0(x)=E_0(x)+F_0(x)\log_{10}x,$$

$$K_1(x)=\frac{E_1(x)}{x}+F_1(x)\log_{10}x.$$

These tables amplify the B.A. Mathematical Tables, Vol. VI, *Bessel Functions, Part I. Functions of Order Zero and Unity* over the range $0\leqslant x\leqslant 1$. In the Introduction it is stated that with the exception of a few entries close to the origin, the tables now reviewed were obtained by interpolation in the B.A. tables.* It is also stated that the last place may be in error by about a unit and in some instances perhaps by two units . . . the ninth significant figure, where given, may be in error by as much as four units. It is difficult to see why the tables were not cut down to at most 8 figures everywhere.

A short Introduction consists mainly of a list of formulae for the four Bessel Functions J, Y, I, K, with a few remarks on the tables themselves and their use.

There are several misprints, particularly in the Introduction—irritating rather than misleading.

(2) This tabulates coefficients $C_i^{(n)}(p)$ in the Lagrange type formula

$$f'(x_0+ph)\sim\frac{1}{hC(n)}\sum_{i=-[\frac{1}{2}(n-1)]}^{i=[\frac{1}{2}n]} C_i^{(n)}(p)f_i,$$

where $[m]$ denotes the largest integer in m, and $f_i=f(x_0+ih)$. Exact values are given for

$$n=4,\quad p=-1(0{\cdot}01)+2,$$

$$n=5,\quad p=-2(0{\cdot}01)+2,$$

$$n=6,\quad p=-2(0{\cdot}01)+3,$$

$$n=7,\quad p=-3(0{\cdot}01)+3.$$

For $n=3$, no tables are given, since the formulae takes the simple form

$$f'(x_0+ph)\sim\frac{1}{h}\,[(p-\tfrac{1}{2})f_{-1}-2pf_0+(p+\tfrac{1}{2})f_1].$$

Both these tables are printed from type, though it is disappointing to see equal height numerals used, rather than the more easily legible old style, head and tail numerals.

* In view of the remarks in the Foreword stating that the B.A. Tables are inaccurate, it seems a little odd that the tables now reviewed should be based on them! In fact, it seems clear that the intention was to refer to *inconvenience* rather than to *inaccuracy*.

(3) These useful tables give

$F(\frac{1}{2}n, \frac{1}{2}; x)$;	$n=3(2)201$, $x=0(0{\cdot}01)0{\cdot}1$ to 7 figures;
$\dfrac{F(\frac{1}{2}n, \frac{1}{2}; x)}{\cosh\sqrt{(2n-1)x}}$;	$n=43(2)201$, $x=0(0{\cdot}01)0{\cdot}1$ to 7 fig. $\equiv$ 6 dec. with leading figures always $1{\cdot}0\ldots$;
$\dfrac{\ln F(\frac{1}{2}n, \frac{1}{2}; x)}{\sqrt{2nx}}$;	$n=3(2)201$, $x=0{\cdot}1(0{\cdot}01)\ 0{\cdot}6(0{\cdot}05$ or $0{\cdot}1)2(0{\cdot}2)7(1)45(5)100$ to 6 dec., values all being within the range $0{\cdot}36$ to $4{\cdot}30$ and concentrated near unity.

A stimulating and interesting introduction by W. Horenstein precedes the tables, giving properties of the function tabulated (many of them seem to be new), including various types of asymptotic expansions, and an expansion in terms of Bessel Functions. Methods of computation and interpolation are also described, and six pages of interpolation charts show which Lagrangian formula should be used in various regions.

Although the Introduction has not been examined critically throughout, it seems worthwhile to point out a few errors that have been located. The use of the sign $\sim$ seems unduly stretched in formula (29). On p. viii, in fact, the ratio $e^{\frac{1}{2}x}\cosh\sqrt{2nx}/\frac{1}{2}e^{\sqrt{2nx}}$ becomes infinite as $x\to\infty$, and $\to e^{\frac{1}{2}x}$ as $n\to\infty$.

Again, on p. xiv, the coefficient of $I_7(x)$ in (70) should be 18 23545 16330/429, and not as given (in two places); secondly, the true value of $F(\frac{41}{2}, \frac{1}{2}; 0{\cdot}2)$ is $30{\cdot}97825\ 032$ to 8 decimals, and this value is given by 13 terms of *either* Bessel function *or* Power series expansion! In fact, the choice of example was unfortunate in that the power series is far easier to compute, and is almost equally convergent.

The recurrence relation (69) for the coefficients c_k in this expansion is not very convenient numerically, since it depends on the process (prohibitively awkward numerically) of replacing α and γ by $\alpha+1$ and $\gamma+1$ in c_k to obtain d_k. It may therefore be worth while to give an alternative. Write

$$c_k = C_k - C_{k+2},$$

then

$$(n+\gamma)C_{n+1}-(n+1+2\alpha-\gamma)C_n+(n+\gamma-2\alpha)C_{n-1}-(n+1-\gamma)C_{n-2}=0,$$

with $C_{-1}=0$, $C_0=1$, $\gamma C_1-2\alpha C_0=0$. Although this has 4 terms, it is simpler to apply.

As is shown by equation (32) in the Introduction, the functions tabulated give Weber functions, or solutions of Weber's equation. In fact,

$$v(y)=e^{-\frac{1}{2}y^2}F(\tfrac{1}{2}n, \tfrac{1}{2}; y^2)$$

satisfies the equation

$$\frac{d^2v}{dy^2}+(1-2n-y^2)v=0.$$

From this equation, a connection with the probability functions $Hh_n(x)$ may be traced. (The $Hh_n(x)$ are tabulated in *B.A. Mathematical Tables*, Vol. I.) The relations are

$$F(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}x^2)=\frac{e^{\frac{1}{4}x^2}}{\sqrt{2\pi}}\{Hh_0(x)+Hh_0(-x)\}=e^{\frac{1}{2}x^2},$$

$$F(\tfrac{3}{2}, \tfrac{1}{2}; \tfrac{1}{2}x^2)=\frac{2e^{\frac{1}{4}x^2}}{\sqrt{2\pi}}\{Hh_2(x)+Hh_2(-x)\}=e^{\frac{1}{4}x^2}h_2^*(x),$$

and generally

$$F\left(\frac{2k+1}{2}, \tfrac{1}{2}; \tfrac{1}{2}x^2\right)=\frac{2^k.k!\,e^{\frac{1}{4}x^2}}{\sqrt{2\pi}}\{Hh_{2k}(x)+Hh_{2k}(-x)\}=\frac{e^{\frac{1}{4}x^2}}{1.3\ldots(2k-1)}h_{2k}^*(x).$$

This identification is less important than it at first appears, since the error *integrals* involved in $Hh_n(x)$, $n \geqslant 0$, are cancelled in the sums

$$Hh_{2k}(x) + Hh_{2k}(-x),$$

leaving only multiples of the Hermite polynomials

$$h^*_{2k}(x) = e^{-\frac{1}{2}x^2} D^{2k} e^{\frac{1}{2}x^2} = (-1)^k h_{2k}(ix).$$

Similar expressions may also be derived in terms of the Hermite polynomials $H^*_{2k}(x) = e^{-x^2} D^{2k} e^{x^2}$.

(4) These tables are rather specialised and very diverse in character ; they appear to have an application only in the particular field for which they were prepared. Only a brief indication of the type of function tabulated will be given. In Part II are given

$$K(m, \alpha), \quad (2n+1)\mathbf{R}(C_n^1) \quad \text{and} \quad (2n+1)\mathbf{R}(C_n^2),$$

where

$$K(m, \alpha) = \frac{2}{\alpha^2} \mathbf{R} \left\{ \sum_{n=1}^{\infty} (2n+1)(C_n' + C_n^2) \right\},$$

in which

$$C_n^1 = \frac{S_n'(\beta) S_n(\alpha) - m S_n'(\alpha) S_n(\beta)}{S_n'(\beta) \phi_n(\alpha) - m \phi_n'(\alpha) S_n(\beta)},$$

$$C_n^2 = \frac{m S_n'(\beta) S_n(\alpha) - S_n'(\alpha) S_n(\beta)}{m S_n'(\beta) \phi_n(\alpha) - \phi_n'(\alpha) S_n(\beta)},$$

$$S_n(x) = \sqrt{\tfrac{1}{2}\pi x}\, J_{n+\frac{1}{2}}(x), \quad C_n(x) = (-1)^n \sqrt{\tfrac{1}{2}\pi x}\, J_{-n-\frac{1}{2}}(x), \quad \phi_n(x) = S_n(x) + iC_n(x).$$

$\beta = m\alpha$, $\alpha = 2\pi r/\lambda$, r being the radius of the spherical particle concerned, and λ the wavelength of the incident light.

Accuracy is usually 4 to 5 figures, values of m vary from 1·33 to 2, α ranges from 0·5 to 5 or 6 (and in Table II to 12).

(5) This is a most welcome table. As remarked in an earlier review by the writer, entitled " The Decimal Subdivision of the Degree " (*Math. Gazette*, **26**, pp. 226–30, 1942), the most convenient decimalisation of angle is that obtained by retaining the familiar nonagesimal degree, and dividing it decimally. This follows the precedent set by Briggs and Gellibrand in *Trigonometria Britannica* in 1633, and rather lost sight of since then. The decimal subdivision seems now to be slowly increasing its popularity, so that the table now reviewed is most opportune.

As with Briggs' table, it gives 15 decimal values of $\sin x$ and $\cos x$ for $x = 0^\circ(0^\circ{\cdot}01)90^\circ$. The arrangement is semi-quadrantal, arguments exceeding 45° being given to the right of the page.

Second differences are given to aid interpolation by means of Everett's formula. These have 8 figures, indicating that linear interpolation in the table is accurate to 8 decimals.

A short introduction is concerned mainly with references to other many-place tables, and with interpolation (direct and inverse) in the tables. Several examples are given.

Altogether this is a useful and handy table. J. C. P. M.

L'Analyse Mathématique. Par André Delachet. Pp. 118. 1949. Collection " Que sais-je? ", 378. (Presses Universitaires de France, Paris)

This is a most attractive outline of the principal ideas in the development of the Calculus, from the work of Fermat to the programme of the new Bourbaki school of mathematics, illustrated by a series of examples as profound and apt as a Grimm's fairy-tale. Only an elementary knowledge of mathematics is assumed, and modern abstract notions are explained in terms of their intuitive geometric content, as in von Koch's construction of a con-

tinuous function which is nowhere differentiable and Buhl's discontinuous solution of an isoperimetric problem.

Judged as a history of mathematicians, *L'Analyse Mathématique* does England scant justice; it would appear to be the French view that, after Newton and Maclaurin, no one in these islands contributed anything to the development of mathematical analysis. The author's bias is not, however, mere nationalism, for the pre-eminence of Gauss and David Hilbert is warmly defended. Fortunately, Delachet's prejudices in no way impair his excellent judgment of the significance of past discoveries and the trend of modern developments. R. L. G.

SECOND AUSTRIAN CONGRESS OF MATHEMATICIANS

An international group of mathematicians representing fourteen different countries held a meeting at the University of Innsbruck in the week August 29th–September 2nd, 1949. The occasion was the second Austrian Congress of Mathematicians. Under the auspices of the Austrian Mathematical Society, the congress, organised by the Society's Vice-President, Professor R. Inzinger, Technische Hochschule, Vienna, was an ambitious sequel to the first congress, held in 1948 and attended only by Austrian mathematicians.

The opening lecture on "Geometry for the mountaineer" was given by Professor L. Vietoris, Innsbruck, as a lecture of general interest to the visitor of the mountain-surrounded city.

Then the mathematicians settled down to more serious lectures and discussions in five different groups: I, Analysis; II, Geometry; III, Algebra and theory of numbers; IV, Applied mathematics; V, History, philosophy and education. It is impossible to give even a short survey of the various lectures revealing the recent developments in research in the various countries. I should only like to mention the lecture by Professor F. Severi (Rome) on "Rigour in algebraic geometry", delivered with a truly Roman enthusiasm and vigour.

The social programme, financed by the Society with the help of the Austrian educational authorities, and assisted by the co-operation and facilities accorded to the Congress by the City of Innsbruck, was most capably organised to allow the visitors to derive the maximum enjoyment from the mountain scenery of the surrounding country and the architectural beauty of Innsbruck, from the local colour of Tyrolean life and the society of fellow members of the congress.

The five days were hectic and strenuous, but they provided a memorable experience, not only from the academic view-point, but also on account of the warm-hearted reception and generous hospitality offered by the Austrians, and by their earnest wish to promote international understanding and co-operation.

E. Stein.

CORRIGENDUM.

Gazette, XXXIII, No. 305 (October 1949), p. 164:

"An extended use of velocity-time graphs." The last sentence of § 3 should be deleted and replaced by "A similar calculation of the relevant areas gives the required result for the distance described".

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